AN INVESTIGATION OF THE STABILITY

OF NONLINEAR FEEDBACK SYSTEMS

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SUMMARY

This dissertation presents several results on the stability of the nonlinear feedback regulator system. The results deal with both stability in the Lagrange sense and global asymptotic stability in the Liapunov sense. The method of analysis is based on an indirect proof which first assumes a steady-state motion that is not identically zero and shows that this assumption leads to a contradiction. This method presents two immediate advantages: the mathematical treatment is rather simple and there are few significant restrictions on the nature of the system. The nonlinearity need not be time-invariant, continuous, or single-valued; and the linear block can be time-varying, distributed, discrete, or may have time-delay and singularity functions in its impulse response. Consequently, the domain of applicability of the results obtained through this method includes a broad class of systems.

In this thesis the method is applied to stationary systems, time-varying systems, sampled-data systems, and multiple-nonlinearity systems. In each case five different results are obtained. The first result describes a sufficient condition that guarantees the Lagrange stability of the system and provides an upper bound on any free steady-state motion that could be sustained by the system. The second result is an improvement criterion that may be used to extend the domain of applicability of any existing criterion of the Popov type to include nonlinearities that cannot otherwise be handled by that criterion. The third result employs a technique of tightening the upper bound on the

system's steady-state motion to predict global asymptotic stability. The fourth and fifth results are theorems of the Popov type which insure global asymptotic stability by requiring the nonlinearity to be confined within a specified region of the input-output plane. A number of examples are given to illustrate the application of the various results. Some of the examples describe specific systems for which the results of this dissertation are shown to be more powerful than the recent results in the literature.

CHAPTER I

THE PROBLEM AND ITS HISTORY

Stability is often important in the analysis of systems studied in various disciplines of science and engineering. Stability theory started as a branch of applied mathematics, but its present uses in such fields as automatic control, network theory, aerospace engineering, and nuclear engineering are practical considerations of prime importance. The results of stability theory find application not only in the analysis of the behavior of systems but in modeling and system design as well. At present, stability theory is one of the outstanding topics of interest in what has become known as modern control theory.

From its beginnings in astronomy and mechanics one can trace a vast body of literature on the subject. What may be surprising is that some of the more interesting results have been discovered quite recently. The problem of stability in the linear, time-invariant system has been thoroughly investigated and the results have been used for several years. Recently, the growing interest in such diverse areas as satellite control, parametric amplifier design, and nuclear engineering has given rise to a class of problems in the stability of nonlinear and time-varying systems. This new class of problems demanded the use of analysis techniques more sophisticated than had been necessary for the linear problem. The relatively higher level of difficulty presented by these problems is not an uncommon feature of most nonlinear analysis. This thesis research takes a new approach in the investigation of the stability of nonlinear and time-varying systems.

Stability Behavior of the Nonlinear Feedback Regulator System

This research deals with the problem of stability of the nonlinear feedback regulator system which has been the object of renewed interest over the past decade. The basic system under consideration is shown below in Figure 1.

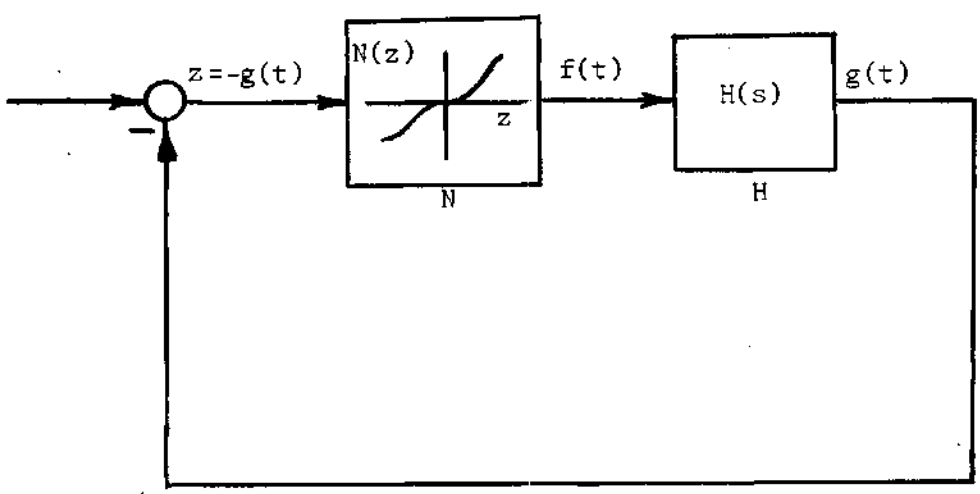

Figure 1. The Nonlinear Feedback Regulator System

The system has the conventional feedback loop with a linear part H represented by its system function H(s) and a nonlinear gain N represented as an input-output characteristic N(z). The time waveform f(t) is the output of the nonlinearity N and input to the linear plant H. The waveform g(t) is the output of H and is fed back negatively as the input z of the nonlinearity. Starting from an arbitrary initial state, the unforced motion of the above system may exhibit one of the following three types of behavior:

1. The system variables may increase in amplitude with no finite upper bound. In such a case, the system would be classified as unstable.

2. The system variables may eventually decrease in amplitude and approach in the steady-state a waveform which is identically zero. Such a system would be classified as asymptotically stable.

3. The system variables may approach sustained waveforms of bounded non-zero amplitudes. An example of this behavior is the phenomenon of a periodic steady-state oscillation known as a *limit cycle*.

The objective of this dissertation is to present new techniques for guaranteeing asymptotic stability for the nonlinear feedback regulator system.

Direct Versus Indirect Proofs of Stability

The stability of the unforced response of the system shown in Figure 1 has been studied analytically using two essentially different methods. The first is the application of the Second Method of Liapunov, and the other is the functional analysis method used in the original work of V. M. Popov which resulted in the well-known Popov Criterion. Both of these methods may be described as employing a direct proof of stability. Each method starts by considering the motion of the system from some arbitrary initial state and proves that if certain conditions are satisfied, the motion will necessarily tend to zero in the steady-state. Thus, stability is verified directly.

The approach followed in this research employs an indirect proof of stability. The first step in the method is to establish sufficient conditions to guarantee that the system motion is bounded. This is followed by the assumption that the bounded motion tends to a steady-state waveform which is not identically zero. This assumption, coupled with certain conditions on the nonlinearity, leads to a contradiction. The resulting contradiction rigorously proves that the starting assumption of a non-zero steady-state trajectory is false, i.e., the motion tends to an identically zero steady-state waveform, and the system is asymptotically stable. This indirect method of proof results in a simpler mathematical treatment than the direct approaches. This simpler treatment is possible because here one may consider the steady-state behavior and avoid the direct consideration of initial conditions. Because of their arbitrariness, the initial conditions complicate the direct proofs and require placing some restrictions on the nature of the nonlinearity to render the analysis less formidable.

The Popov Theorem and Its Applicability

The Popov criterion has been in many respects the central and most important general result. It formulates a sufficient condition for the stability of the system in terms of confining the nonlinearity $N(z)$ to some sector of the input-output plane, the magnitude of the sector being determined from the frequency response of the linear plant. Indeed, the value and importance of the Popov criterion both as a theoretical result and as an analysis and design criterion lies largely in its simplicity, effectiveness, and degree of generality. Yet the

Popov criterion is not completely general in its applicability. The original Popov theorem is proved for a system with specific restrictions placed on the linear and nonlinear portions. The nonlinearity is stipulated to be continuous, memoryless, and time-invariant. The linear plant is required to be continuous, time-invariant, and finite dimensional. Furthermore, it must have no delayed arguments, and must be representable by a system function H(s) whose numerator has a degree less than the degree of its denominator, i.e., no impulses or doublets are permitted in the impulse response of the linear plant. Several extensions have been successfully performed to remove some of these restrictions, but no available extension handles the simultaneous removal of all the above restrictions in a single system.

The Applicability of the Thesis Results

The method and criteria presented in this thesis represent a step in that direction. The degree of generality afforded by this approach is such that the results can be applied to a system which exhibits simultaneously a number of complicating features. The nonlinearity can have discontinuities and memory (hysteresis) and can be time-varying. The linear plant can be discrete, distributed, time-varying, and can have delayed arguments and singularity functions in its impulse response. The method and results are also applicable to systems containing several nonlinear characteristics.

Another general result offers some improvement over any given criterion of the Popov type or any of its extensions and refinements. This improvement is in the sense of relaxing the restrictions placed on the nonlinearity by the given criterion over some portion of the input axis. Specifically, this improvement theorem (Theorem II in Chapter III) allows the nonlinearity to leave and remain outside the stability sector indicated by the given criterion. Thus, one is able to predict stability for some nonlinearities that lie in a sector larger than, for example, the Popov sector.

Because these results can handle nonlinearities that exhibit discontinuities and hysteresis, they lend themselves directly to the study of the phenomenon of limit cycles (sustained unforced oscillations) in a large class of regulator systems that have been traditionally analyzed by such methods as the describing function, the Tsypkin locus, the Hamel locus, and the phase-plane plot, all of which have certain shortcomings and are not always applicable. The describing function technique is approximate and, therefore, may give false indications. The Tsypkin and Hamel techniques are only applicable to the simplest cases of relay-type nonlinear characteristics. The phase-plane method is specifically intended for second-order systems. These methods cannot be applied to time-varying nonlinearities, and in general, they are more effective as analysis tools rather than design criteria, especially when compared to a Popov-type criterion which formulates the conditions for stability in terms of confining the nonlinearity to a sector rather than making specific references to any detailed features of the nonlinearity. A Popov-type criterion would be more powerful from the design standpoint and may considerably reduce the design effort.

The History of the Problem

The first part of this section gives the various definitions of stability that are relevant to the problem. The second part describes briefly the main techniques of investigating stability: the Liapunov method and the Popov method. The last part gives a summary of the more significant results that have been developed since the discovery of the Popov criterion.

Definitions of Stability

The connotation of the term *stability* is not unique. Investigators have found it convenient to formulate and study different concepts of stability because each of the different notions had definite practical implications. For a given system, one notion of stability may be more pertinent than the others, depending upon the desired system performance.

The various definitions of interest here may be categorized under two broader notions, namely, the concept of Liapunov stability and the concept of Lagrange stability. Liapunov stability is concerned with the behavior of the motion of the system in a sufficiently small region around a given equilibrium state. Lagrange stability, on the other hand, deals with the eventual boundedness of the trajectories of motion of the system without regard to any equilibrium point or any neighborhood thereof. The precise definitions using the state variable description of the system are given in the following paragraphs.

Liapunov Stability. The dynamic relationships of the system shown in Figure 1 can be represented by the general unforced vector differential equation:

$$\dot{\underline{x}} = \underline{f}(\underline{x},t)$$

where $\underline{x}$ and $\underline{f}$ are n-dimensional column vectors. The state $\underline{x}_e$ is said to be an equilibrium state if

$$\dot{\underline{x}}\big|_{\underline{x}=\underline{x}_e} = \underline{f}(\underline{x}_e,t) \equiv \underline{0}$$

Definition 1 (Liapunov Stability): The system is said to be stable in the sense of Liapunov with respect to the equilibrium point $\underline{x}_e$ if for any given $\varepsilon > 0$ there exists another positive real number, $\delta(\varepsilon,t_o)$, such that for every initial state satisfying the inequality $\|\underline{x}(t_o)-\underline{x}_e\| < \delta$, the resulting trajectory satisfies the inequality, $\|\underline{x}(t)-\underline{x}_e\| < \varepsilon$, for all $t \geq t_o$.

Definition 2 (Local Asymptotic Stability): The system is said to be asymptotically stable in a region D of the state space if

$$\lim_{t\to\infty}\|\underline{x}(t)-\underline{x}_e\| = 0$$

whenever

$$\underline{x}(t_o) \in D$$

Definition 3 (Global Asymptotic Stability): The system is said to be globally asymptotically stable, i.e., asymptotically stable in the large, if the region D of local asymptotic stability extends over the entire state space.

Lagrange Stability. The notion of Lagrange stability deals with the eventual boundedness of the motion trajectory starting from a certain initial state. It is not concerned with the behavior of the motion around any specific equilibrium state of the system.

Definition 1 (Lagrange Stability): The system is said to be bounded, or stable in the Lagrange sense, if for every bounded region V in state space there exists another bounded region B such that any trajectory starting in V does not leave B for all $t > t_o$:

$$\underline{x}(t) \in B \quad \text{for all } t > t_o$$

whenever

$$\underline{x}(t_o) \in V$$

Definition 2 (Asymptotic Lagrange Stability): The system is said to be ultimately bounded, or asymptotically stable in the Lagrange sense, if a bounded region U can be found such that the trajectory of motion will eventually enter U for any bounded initial state $\underline{x}(t_o)$, i.e., for any $\underline{x}(t_o)$ there exists an instant $t_1 \geq t_o$ such that

$$\underline{x}(t) \in U \quad \text{for all } t \geq t_1$$

The Methods of Investigating Asymptotic Stability

The techniques that have been used to study the stability of systems may be grouped into two categories:

1. Those methods that deduce stability by actually solving for the trajectory of motion, or at least by deducing the functional form of the trajectory.

2. Those methods that do not solve for the actual trajectory but place sufficient conditions on the system to guarantee that the motion will tend asymptotically (as $t \to \infty$) to the equilibrium state.

In the first category, one may include the methods of stability analysis for linear systems such as the Routh-Hurwitz Criterion or the Nyquist Criterion. These methods, in effect, prove stability by showing that the functional form of the system trajectories is a linear combination of exponentially decaying waveforms. Therefore, these methods give criteria which are necessary conditions as well as sufficient. The methods of Liapunov theory and those used in the Popov criterion and most other recent investigations fall in the second category. These are useful in nonlinear systems where direct analytical solution of the system trajectories is seldom possible.

The use of Liapunov theory [1-3] marks a turning point in the study of stability. The basic principles of the Liapunov methods were known before the turn of the century but its application to nonlinear stability problems was started in the early fifties. The basic idea in the Liapunov method is to find a suitable scalar function of the state variables $V(\underline{x})$ which together with its time derivative should satisfy certain conditions of sign definiteness. The success of the method in any particular situation is contingent upon the availability of such a Liapunov function, and the original theory of Liapunov does not indicate any systematic methods for finding a suitable $V(\underline{x})$. At present, several successful investigations have led to systematized ways for finding $V(\underline{x})$ for certain classes of systems, but this difficulty has not yet been solved in general. Thus, in many problems the nature of an adequate Liapunov function depends on the particular details of the system. Finding such a function might require a considerable amount of ingenuity, time, and effort on the part of the system analyst.

The Popov Criterion [7] has partially overcome this disadvantage. This important criterion, which was developed in 1961 by V. M. Popov through the use of both functional analysis techniques and Liapunov theory, is applicable to a broad class of systems. The Popov theorem states that the system shown in Figure 1 is asymptotically stable in the large if the nonlinearity $N(z)$ is confined to a sector bounded by the z-axis and the line kz, where k is determined from the inequality

$$\frac{1}{k} + \mathrm{Re}\{(1+j\beta\omega)H(j\omega)\} \geq 0 \qquad \text{for all real } \omega$$

and β is an arbitrary real scalar constant. The Popov theorem is described as a frequency domain criterion because its final statement is expressed in terms of the frequency domain description of the linear plant. This description has a simple graphical interpretation using a frequency domain plot which is a modification of the familiar Nyquist plot. The simplicity and generality of the Popov Criterion make it a very effective tool both in analysis and design.

Later Developments and Results

The discovery of the Popov Criterion gave a new impetus to the problem of stability, and several important investigations were pursued in the same direction which produced several new results and extensions to systems not covered by the original Popov theorem. The more significant results found in the past seven years are:

1. Additional frequency domain criteria [8-16], and "improved" Popov-type criteria attained at the expense of placing more restrictions on the nature of the nonlinearity [17-20].

2. Stability criteria dealing with systems that contain a time-varying nonlinear characteristic [23-26].

3. Criteria applicable to nonlinear sampled-data systems [29-34].

4. Extensions of previous results to systems containing more than one nonlinear characteristic [21-22].

5. Extensions to distributed-parameter systems [13-16].

Like the Popov Criterion, all of these investigations utilized a direct proof, while the approach of this thesis research involves an indirect method of proof.

Outline of the Thesis

The second chapter is devoted to a presentation of the general method of approach and its applicability. A specific outline of the method of proof by contradiction is given. Chapter III describes in detail the application of the method to stationary systems. Five different theorems with their complete proofs are given. The fourth chapter deals with time-varying systems in which the plant, the nonlinearity, or both can be time-varying. Five new results, analogous to those of the stationary system, are given. In Chapter V the results are extended to sampled-data systems. In the sixth chapter a system with several nonlinearities is examined and corresponding results are found. Conclusions and recommendations for further investigation are presented in Chapter VII.

CHAPTER II

THE METHOD AND ITS APPLICABILITY

Equivalent Representations

In this investigation an equivalence transformation is applied to a given feedback system [4]. The transformation enables one to obtain for the system a set of different representations which have equivalent stability properties. Consequently, the stability of the given system may be investigated by using any of its equivalent representations. The steps in the transformation are shown in Figure 2(a,b,c,d). Figure 2(a) represents the given basic feedback system. In Figure 2(b) the given nonlinear characteristic $N_o(z)$ has been resolved into a parallel combination of two characteristics

$$N_o(z) \equiv N_k(z) + kz \tag{2.1}$$

where k is any finite real scalar constant. In Figure 2(c) the linear gain block is redrawn to show its relationship to $H_o(s)$. Obviously, the original plant transfer function and the linear feedback gain k form a linear feedback loop which may be replaced by its equivalent block $H_k(s)$:

$$H_k(s) = \frac{H_o(s)}{1 + kH_o(s)} \tag{2.2}$$

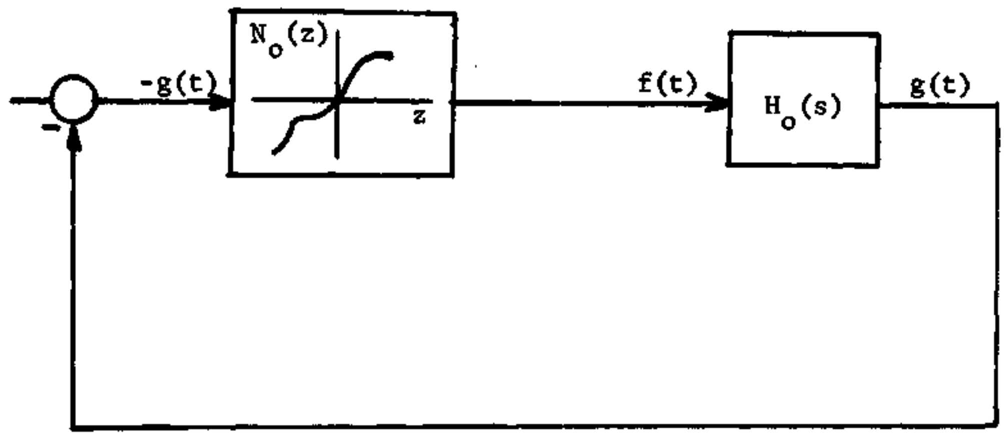

Figure 2(a). The Original Representation $R_o(N_o,H_o)$

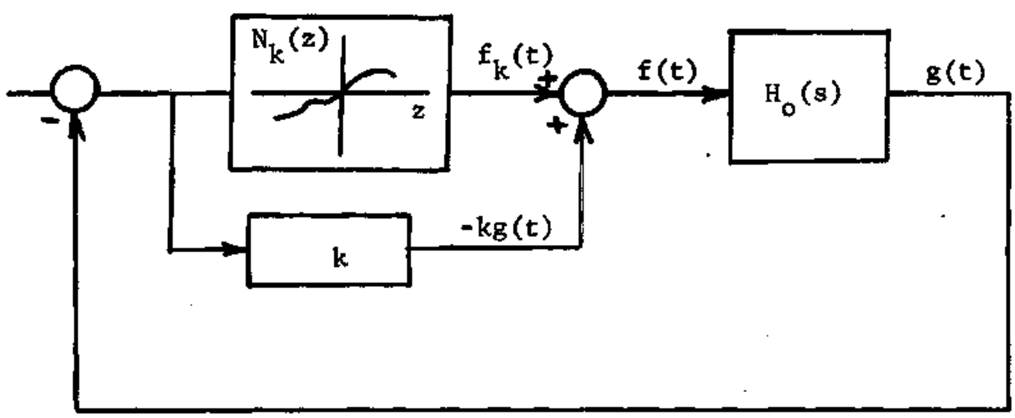

Figure 2(b). Decomposition of $N_o(z)$

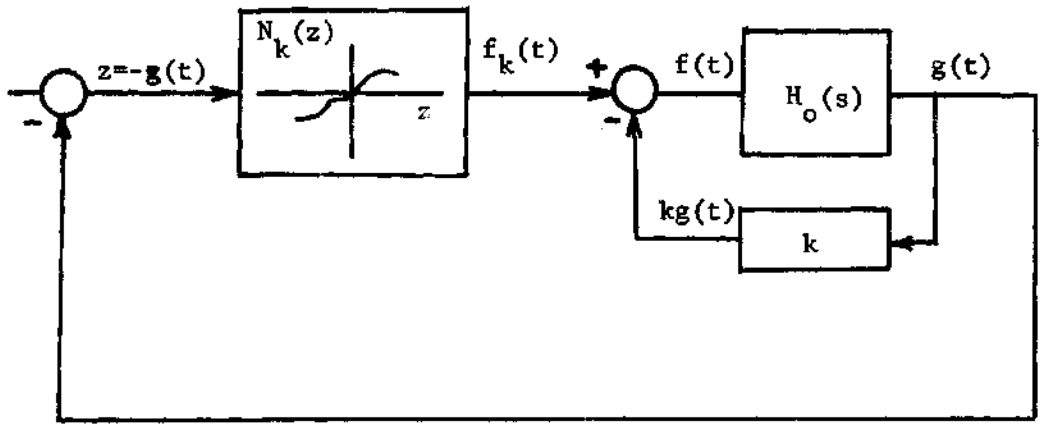

Figure 2(c). Relationship of the Gain k to $H_o(s)$

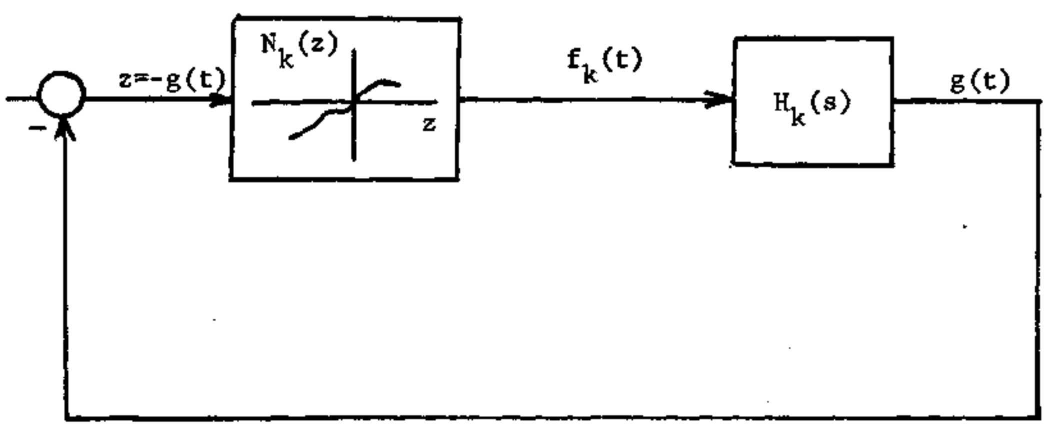

Figure 2(d). The Equivalent Representation $R_k(N_k, H_k)$

The resulting system is shown in Figure 2(d) and has the same form as the original system in Figure 2(a) but with different linear and nonlinear blocks. The representations (a) and (d) are equivalent with respect to g(t).

The representation of the original system, as in Figure 2(a), is denoted by $R_o(N_o,H_o)$ and any equivalent representation corresponding to a specific value of k, as in Figure 2(d), is denoted by $R_k(N_k,H_k)$.

The Conceptual Steps of the Method

The stability of the given system is investigated by examining the behavior of the waveform g(t). Starting with an arbitrary initial state, the system is globally asymptotically stable if

$$\lim_{t\to\infty} g(t) = 0 \tag{2.3}$$

Since g(t) is independent of the particular representation $R_k(N_k,H_k)$ of the given system, the analysis is not restricted to the consideration of a specific representation. One may examine the entire ensemble of representations denoted by the set

$$\{R_k\} \qquad -\infty < k < \infty \tag{2.4}$$

Consider the representation in Figure 2(d) and let g(t) be expressed in terms of the input to the linear plant $f_k(t)$ and any available characterization of the linear plant $H_k(s)$, such as its unit impulse response $h_k(t)$:

$$g(t) = [g(t-t_0)]_{\substack{k \\ trans}} + \int_{t_o}^{t} f_k(x)h_k(t-x)dx \qquad (2.5)$$

Note that both the transient and the forced components depend upon the particular representation R_k but their sum is equal to g(t) for any choice of k.

The method for proving the stability of the system may be stated in the following conceptual steps:

(i) Examine g(t) as $t \to \infty$ or equivalently as $t_o \to -\infty$.

(ii) Establish sufficient conditions such that

$$\lim_{t_o \to -\infty} [g(t-t_0)]_k = 0 \qquad (2.6)$$

(iii) Allow the system sufficient time for the transient component to die out and examine the resulting waveform g(t) which is now equal to the steady-state component, i.e., as $t \to \infty$ or $t_o \to -\infty$,

$$g(t) = [g(t)]_{\substack{k \\ ss}} = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx \qquad (2.7)$$

The waveform $[g(t)]_{\substack{k \\ ss}}$ may possibly be an unbounded function of time.*

(iv) Establish sufficient conditions for the system so that g(t) would be a bounded function (Lagrange stability).

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*The function $f_k(x)$ in 2.7) is the input to the linear plant appearing in (2.5) when $t_o = -\infty$. The integrand in (2.7) is assumed to be Riemann integrable.

(v) Assume that the bounded steady-state function g(t) is not identically zero, i.e.,

$$[g_{ss}(t)]_k \not\equiv 0 \tag{2.8}$$

(vi) Impose certain conditions on the linear and nonlinear portions of the system.

(vii) Use the conditions in (vi) and the assumption in (v) to deduce a contradiction. This contradiction implies that if the conditions in (vi) are satisfied the assumption in (v) cannot be true. Thus, $g_{ss}(t) \equiv 0$, and the system is asymptotically stable in the large. The conditions in (vi) are, therefore, sufficient conditions for stability.

The sufficient condition for the asymptotic decay of $[g_{trans}(t-t_0)]_k$ stated in (ii) is easily recognized. Since $[g_{trans}(t-t_0)]_k$ is the unforced response of the linear system whose transfer function is $H_k(s) = H_o/(1+kH_o)$, it is sufficient to require that the poles of $H_k(s)$ be strictly in the left half plane. This requirement implies that the value of k must belong to the set of stable gains for the single loop linear feedback system having $H_o(s)$ as the plant. This set will be denoted by $I = \{k_{stable}\}_{H_o}$. Thus, the condition in (ii) is satisfied if

$$k \in I \tag{2.9}$$

The set I can be determined by any of the well-known methods of linear control theory, such as the Routh-Hurwitz Criterion, the Nyquist Criterion, or the root locus technique. Typically, the set I is an open interval $(k_{RH-},k_{RH+})_{H_o}$, where the values k_{RH-}, k_{RH+} are the limits of the Routh-Hurwitz stability sector for the plant $H_o(s)$. These limits must be excluded from the set I because they result in a transient response which does not decay with time. In some cases, the set I may consist of more than one simple interval of values, as in conditionally stable systems. It should be noted that the condition (2.9) does not imply that the given plant H_o must be stable. In many cases where $H_o(s)$ has poles in the right half plane one can find representations $R_k(N_k,H_k)$ such that $H_k(s)$ has all of its poles in the left half plane.

Applicability of the Method

The method of proof outlined in the previous section places no significant restrictions on the nature of the nonlinearity or the linear plant. Consequently, the results obtained through this method have a broad domain of applicability. It is shown in this investigation that the method can be applied to a system that has any combination of the following descriptions:

(i) Features of the nonlinearity. The nonlinearity may be discontinuous at any number of points other than the origin. It may have memory in the form of hysteresis. It may be time-varying, and the time variation need not be continuous. The method does not stipulate that the nonlinearity be restricted to any prescribed quadrants of the input-output plane. Thus, the graph of $N_o(z)$ or $N_k(z)$ may lie in all four quadrants.

(ii) Characteristics of the linear plant. The linear system may be lumped or distributed, stationary or time-varying, differential or nondifferential, and causal or anticipatory. It may exhibit delayed arguments such as transportation lag in the feedback loop. The system function may be expressible as a rational fraction, $H(s) = N(s)/D(s)$, in which the numerator has a degree equal to or one higher than the degree of the denominator, which corresponds to an impulse response function having an impulse or a doublet, respectively.

(iii) Time operation. The type of signals in the system may be discrete or continuous or both. Thus, sampled-data system may also be handled by the method.

(iv) Complexity of the system. In addition to the simple configuration shown in Figure 1, the method can be extended to systems consisting of any number of cascaded nonlinearities alternating with linear blocks.

It should be noted that the method is capable of handling a system that exhibits simultaneously any number of the above-mentioned properties.

General Description of the Results

Each of the following chapters deals with a different type of system. Specifically, Chapters III-VI treat stability in the stationary system, the time-varying system, the sampled-data system, and the multiple nonlinearity system in that particular order. In each case the main results are stated as theorems, and the corresponding theorems that are of a similar nature are given the same Roman numeral designation in all chapters. For example, Theorems I, I.a, I.b, I.c, and I.d are concerned with the Lagrange stability of their respective systems, and will be referred to collectively as *Theorems I.*

The general results developed in this dissertation may be described as follows:

(a) A criterion that guarantees the Lagrange stability of the motion of the system and provides an upper estimate on the amplitude of any free sustained oscillation that might develop in the system [Theorems I and Corollaries].

(b) An improvement criterion that extends the domain of applicability of any existing criterion of the Popov type to include nonlinearities that otherwise could not be handled by that criterion [Theorems II].

(c) A criterion that can verify global asymptotic stability by an iteration procedure which progressively tightens the bound on the steady-state motion [Theorems III].

(d) Three different results of the Popov type that guarantee global asumptotic stability by requiring the nonlinearity to be confined to a specified region of the input-output plane [Theorems IV, Theorems V(i), and Theorems V(ii)].

In the following chapter the results are developed for the stationary system. The proofs of the theorems are presented in complete detail. In the theorems of subsequent chapters the analogous proofs are given with more concern for brevity.

CHAPTER III

STATIONARY SYSTEMS

In this chapter the general results of the research will be stated and proved for the regulator system shown in Figure 1. It is assumed that both the nonlinearity N(z) and the linear plant H(s) are time-invariant. The problem of the boundedness of the steady-state motion of the system is considered in the first section of the chapter. Theorem I gives sufficient conditions for the Lagrange stability of the system and an upper bound on the motion is found. In the second section an improvement criterion is presented as Theorem II. The last section deals with global asymptotic stability. Three different results are stated in Theorems III, IV, and V. A number of examples at the end of the chapter serve to illustrate the significance of the theorems and their application. In the examples, the thesis results are compared to some of the recent results on the subject.

Lagrange Stability

In this section a sufficient condition for the boundedness of the steady-state motion will be given. A first approximation to the upper bound on the amplitude of the steady-state waveform will be found. The knowledge of this bound will then be used to find progressively tighter bounds.

Theorem I

In the nonlinear regulator system the input and output waveforms of the linear plant resulting from finite arbitrary initial conditions are bounded if there exist constants b_1, b_2, b_3, b_4, λ_1, λ_2, C, and k such that:

(i) $b_1 \geq b_2$, $b_3 \geq b_4$, $\lambda_1 \geq \lambda_2$, $C > 0$

(ii) $k \varepsilon I$

(iii) $b_2 + kz \leq N_o(z) \leq b_1 + kz$ for all $z > \lambda_1$

$b_4 + kz \leq N_o(z) \leq b_3 + kz$ for all $z < \lambda_2$

$|N_o(z)| \leq C$ for all $\lambda_2 \leq z \leq \lambda_1$.

An illustration of a nonlinearity that satisfies the conditions of the theorem is shown in Figure 3. It should be noted that the conditions stated in Theorem I allow the nonlinearity a great degree of flexibility. No constraint is placed on the magnitude of the constants b_1, b_2, b_3, b_4, λ_1, λ_2, C; and their signs can be positive or negative. The conditions of the theorem imply that the nonlinearity must eventually enter and stay within the stability sector of the linear system, i.e., the Routh-Hurwitz sector.

Proof of Theorem I. Consider the representation $R_k(N_k,H_k)$ of the given system in Figure 1, where

$$N_o(z) \equiv N_k(z) + kz \tag{3.1}$$

The first part of the proof is to show that $N_k(z)$ is bounded. From

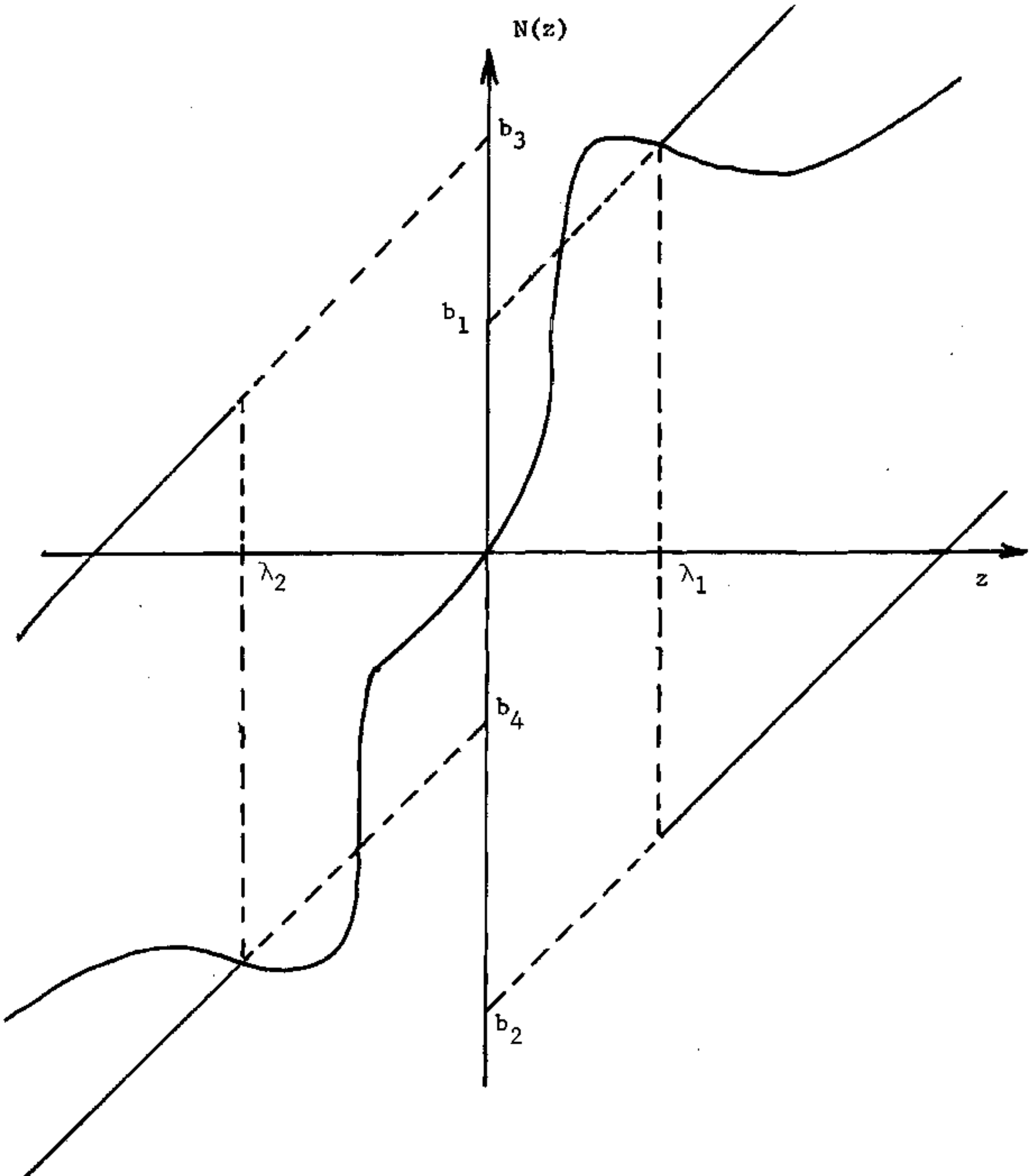

Figure 3. An Illustration of a Nonlinearity Satisfying Theorem I.

hypothesis (iii) on $N_o(z)$ it follows that $N_k(z)$ satisfies the following relationships:

$$b_2 + kz \leq N_k(z) + kz \leq b_1 + kz \qquad \text{for all } z > \lambda_1$$

$$b_4 + kz \leq N_k(z) + kz \leq b_3 + kz \qquad \text{for all } z < \lambda_2$$

$$|N_k(z) + Kz| \leq C \qquad \text{for all } \lambda_2 \leq z \leq \lambda_1$$

These relationships may be rewritten as:

$$b_2 \leq N_k(z) \leq b_1 \qquad \text{for all } z > \lambda_1 \tag{3.2}$$

$$b_4 \leq N_k(z) \leq b_3 \qquad \text{for all } z < \lambda_2 \tag{3.3}$$

$$-C-k\lambda_1 \leq N_k(z) \leq C - k\lambda_2 \qquad \text{for all } \lambda_2 \leq z \leq \lambda_1 \tag{3.4}$$

Furthermore, (3.2), (3.3), and (3.4) may be simplified to:

$$N_k(z) \leq \max\{|b_1|,|b_2|\} \qquad \text{for all } z > \lambda_1 \tag{3.5}$$

$$N_k(z) \leq \max\{|b_3|,|b_4|\} \qquad \text{for all } z < \lambda_2 \tag{3.6}$$

$$|N_k(z)| \leq \max\{|-C-k\lambda_1|,|C-k\lambda_2|\} \qquad \text{for all } \lambda_2 \leq z \leq \lambda_1 \tag{3.7}$$

Define the positive number M as

$$M \triangleq \max\{|b_1|,|b_2|,|b_3|,|b_4|,|-C-k\lambda_2|,|C-k\lambda_1|\}$$

and observe that (3.5), (3.6), and (3.7) imply that

$$|N_k(z)| \leq M \quad \text{for all } z \tag{3.8}$$

Therefore, the nonlinearity $N_k(z)$ is bounded by the quantity M in the representation $R_k(N_k,H_k)$.

The hypothesis in (ii) implies that the linear system function

$$H_k(s) = \frac{H_o(s)}{1 + kH_o(s)}$$

has its poles in the left half plane, and consequently, the impulse response $h_k(t)$ is composed of a linear combination of decaying time functions of exponential order. Hence, the improper integral

$$\int_o^\infty |h_k(t)|dt$$

exists and will be denoted by A_k, i.e.,

$$A_k \triangleq \int_o^\infty |h_k(t)|dt$$

Now consider the output of the linear plant H_k in the representation $R_k(N_k,H_k)$:

$$g(t) = [g(t-t_0)]_{\substack{k \\ trans}} + \int_{t_o}^{t} f_k(x)h(t-x)dx$$

$$|g(t)| \leq |[g(t-t_0)]_{\substack{k \\ trans}}| + |\int_{t_o}^{t} f_k(x)h(t-x)dx| \tag{3.9}$$

where the function $f_k(x)$ is the output of the nonlinearity. Thus,

$$f_k(x) = N_k(-g(x)) = N_k(z(x))$$

From (3.8), one has

$$|f_k(x)| = |N_k(z)| \leq M \tag{3.10}$$

The inequality in (3.9) can be further written as

$$|g(t)| \leq |[g(t-t_0)]_{\substack{k \\ trans}}| + \int_{o}^{t} |f_k(x)||h(t-x)|dx$$

$$|g(t)| \leq |[g(t-t_0)]_{\substack{k \\ trans}}| + \max|f_k(x)| \int_{o}^{t} |h(t-x)|dx$$

But from (3.10), one has

$$\max|f_k(x)| \leq M$$

and

$$|g(t)| \leq |[g(t-t_0)]_{\substack{k \\ trans}}| + M \int_{t_o}^{t} |h_k(t-x)|dx$$

Examining the steady-state behavior g(t) by letting $t \to \infty$, or equivalently by letting $t_o \to -\infty$, one obtains

$$\lim_{t_o\to-\infty} |g(t)| \leq \lim_{t_o\to-\infty} |[g(t-t_0)]_{\substack{k \\ \text{trans}}}| + \lim_{t_o\to-\infty} M\int_{t_o}^{t} |h_k(t-x)|dx$$

Since $k \in I$, the first term on the right side is identically zero and the resulting expression becomes

$$\lim_{t_o\to-\infty} |g(t)| \leq M \lim_{t_o\to-\infty} \int_{t_o}^{t} |h_k(t-x)|dx \tag{3.11}$$

In (3.11), introduce a change of variable by letting

$$t - x = y$$

$$\int_{t_o}^{t} |h_k(t-x)|dx = -\int_{t-t_o}^{o} |h_k(y)|dy = \int_{o}^{t-t_o} |h_k(y)|dy$$

Relationship (3.11) takes the form:

$$\lim_{t_o\to-\infty} |g(t)| \leq M \lim_{t_o\to-\infty} \int_{o}^{t-t_o} |h_k(y)|dy$$

$$\leq M \int_{o}^{\infty} |h_k(y)|dy = MA_k$$

Therefore,

$$\lim_{t_o\to-\infty} |g(t)| \leq MA_k \tag{3.12}$$

The result in (3.12) implies that in the steady-state the function $\underset{ss}{g(t)}$ is bounded, which completes the proof of the theorem.

Corollary. If the system satisfies the conditions of Theorem I, the steady-state waveform $\underset{ss}{g(t)}$ is bounded by

$$|\underset{ss}{g(t)}| \le MA_k = B_1 \tag{3.13}$$

This follows immediately from the result stated in (3.12), where $\lim_{t_0 \to -\infty} g(t)$ is identified as being the steady-state waveform $\underset{ss}{g(t)}$.

Iteration of the Bound

It is assumed that the nonlinearity $N_o(z)$ is given and the bound B_1 on $\underset{ss}{g(t)}$ has been determined. The knowledge of B_1 and $N_o(z)$ may be used to obtain a possibly tighter bound B_2 on $\underset{ss}{g(t)}$, which in turn may be used to yield a still tighter bound B_3. The process is repeated, and the nonincreasing sequence B_i must have a greatest lower bound, i.e.

$$B_1 \ge B_2 \ge \cdots \ge B_n \cdots \ge B_{min} \triangleq \text{glb}\{B_i\}$$

The value B_{min} will be referred to as the iterated bound. The process of tightening the bound is described and justified in the remainder of this section.

Since the steady-state output $\underset{ss}{g(t)}$ of the linear block is independent of the particular representation R_k, one has

$$\underset{ss}{g(t)} = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx$$

which is valid for any $k \varepsilon I$. Furthermore,

$$\left|g(t)\right|_{ss} \leq \max_t \left|f_k(t)\right| \int_{-\infty}^{t} \left|h_k(t-x)\right| dx \tag{3.14}$$

Moreover, because $f_k(t)$ is the output of $N_k(z)$, one may write

$$\max_t \left|f_k(t)\right| \leq \max_z \left|N_k(z)\right| \tag{3.15}$$

where

$$z = -g(t)$$

From (3.13)

$$\left|g(t)\right|_{ss} = \left|z(t)\right|_{ss} \leq B_1$$

Thus, the maximization over z indicated in (3.15) need not be performed over all values of z, but only over those values of z in the interval $(-B_1, B_1)$. Thus, define

$$M_{k.B_1} = \max_{-B_1<z<B_1} \left|N(z)\right|_k \tag{3.16}$$

Relation (3.14) may be written as

$$\left|g(t)\right|_{ss} \leq M_{k.B_1} A_k \tag{3.17}$$

where

$$A_k = \int_{-\infty}^{t} |h_k(t-x)|dx = \int_{0}^{\infty} |h_k(y)|dy$$

Since the inequality (3.17) holds for any $K \in I$, it must be true for the particular value of K that minimizes the right side of (3.17), i.e.,

$$|g(t)|_{ss} \leq \min_{k\epsilon I}\{M_{k.B_1} \cdot A_k\} \tag{3.18}$$

Define

$$B_2 = \min_{k\epsilon I}\{M_{k.B_1} \cdot A_k\}$$

Therefore, relation (3.18) takes the form

$$|g(t)|_{ss} \leq B_2 \tag{3.20}$$

If the bound B_2 is less than B_1, the above procedure may be repeated to obtain B_3:

$$M_{k \cdot B_2} = \max_{-B_2<z<B_2} |N_k(z)|$$

$$B_3 = \min_{k}\{M_{k \cdot B_2} \cdot A_k\}$$

Similarly, one can obtain B_4, B_5, ..., B_{min}. The steps involved in the determination of B_{min} may be implemented in the following manner:

1. Determine $M_{k\cdot B_1}$ as a function of k. This may be recorded as a graphical plot of $M_{k\cdot B_1}$ versus k. From (3.16),

$$M_{k\cdot B_1} = \max_{-B_1<z<B_1} |N_k(z)| = \max_{-B_1<z<B_1} |N_o(z)-kz|$$

Thus, for each k the value of $M_{k\cdot B_1}$ is determined as the maximum deviation of the given nonlinearity $N_o(z)$ from the straight line kz. The maximization may be performed analytically or graphically depending on the form in which $N_o(z)$ is specified.

2. Determine a plot of A_k versus K. This may be obtained analytically or through the aid of a digital or analog computer.

3. From the results of steps 1 and 2, obtain a plot of $M_{k\cdot B_1} \cdot A_k$ versus k.

4. From the plot in step 3, read

$$B_2 = \min_k \{M_{k\cdot B_1} \cdot A_k\}$$

5. Repeat steps 1 through 4 until B_{min} is obtained.

The bound B_{min} is an upper estimate of the amplitude of any sustained steady-state motion that may develop in the system. This bound may be useful in evaluating the possible harmful effects of a limit cycle on the given system.

An Improvement Criterion

In this section it is shown that the knowledge of the bound B_{min} can be used to improve the effectiveness and domain of applicability of any stability criterion whose conditions are stated in terms of the characteristics of $N_o(z)$ in the input-output plane, such as the Popov theorem or any of its variations. The stability criterion to be improved will be referred to as the "given criterion." In the following theorem the improvement criterion will be stated and proved for the case where the given criterion is a theorem on the global asymptotic stability of the system. For other cases the proof is quite similar.

Theorem II

Given any criterion which guarantees global asymptotic stability, the system is globally asymptotically stable if it satisfies the hypothesis of the given criterion within the interval $-B_{min} \leq z \leq B_{min}$. Outside the interval $[-B_{min}, B_{min}]$, the nonlinearity need not satisfy any of the conditions of the given criterion and is, therefore, not restricted by any conditions other than those of Lagrange stability (Theorem I).

Proof (by Contradiction). Assume that the given nonlinearity $N_o(z)$ satisfies the requirements of the given criterion over the interval $[-B_{min}, B_{min}]$ but violates these requirements outside the interval. Furthermore, assume that the system can admit a non-zero steady state solution. It is to be shown that this second assumption leads to a contradiction.

Since the system satisfies the conditions of Theorem I, the steady-state waveform is bounded, i.e.

$$-B_{min} \leq g(t)_{ss} \leq B_{min}$$

After allowing the system sufficient time to settle to its steady-state motion, one can always carry out the following conceptual experiment. Replace the portion of N(z) outside the interval $[-B_{min}, B_{min}]$ by any other characteristic such that the new nonlinearity thus constructed would satisfy over all z the given conditions of stability. This modification of $N_o(z)$ should not affect in any way the steady-state motion occurring within $[-B_{min}, B_{min}]$ because the dynamics of the steady-state motion are controlled entirely by the portion of $N_o(z)$ over $[-B_{min}, B_{min}]$. With the new nonlinearity one has, in effect, a system satisfying all the conditions of the given criterion and yet exhibiting a non-zero steady-state motion. This contradicts the fact that any system satisfying the hypothesis of the given criterion must be asymptotically stable in the large. Therefore, the starting assumption, namely, the existence of a non-zero steady solution, must be false and the system must be asymptotically stable.

Remarks

1. The above theorem is a partial relaxation of the restrictions on the system demanded by the given stability criterion. Thus, the theorem is evidently an improvement over the given criterion. As an analysis tool it allows one to predict stability for certain systems which the given criterion cannot handle. As a design tool it allows more flexibility on N(z) by relaxing the restrictions of the criterion on N(z) outside $[-B_{min}, B_{min}]$. For instance, taking the Popov Theorem as the "given criterion," one may be able to predict stability for a

system with a nonlinearity that does not lie entirely inside the Popov sector or that exhibits jumps or hysteresis at some points outside the interval $[-B_{min}, B_{min}]$.

2. The improvement criterion can be used to render more effective the theorems which follow in this chapter, and for that matter, to improve any relevant new stability criteria that will be developed in the future.

Global Asymptotic Stability

In this section four results on asymptotic stability are presented. The results are sufficient conditions that guarantee an asymptotically stable motion starting from arbitrary initial conditions. The first theorem is a direct consequence of the results on Lagrange stability. The second theorem specifies certain symmetrical sectors of the input-output plane as stability regions. In the third theorem, two results are presented in which the stability regions are not symmetrical sectors.

Theorem III

If $B_{min} = 0$ or if $\mathrm{glb}\{B_n\} = 0$, then the system is globally asymptotically stable.

The validity of this assertion follows from the discussion on the iteration of the steady-state bound. If $B_{min} = 0$, the amplitude of $|g(t)|_{ss}$ is also zero. If, on the other hand, the process of iteration does not terminate but results in a sequence of positive numbers, $\{B_n\}$, whose greatest lower bound is zero, then $|g(t)|_{ss}$ must be identically zero. If this were not true then one would have

$$|g(t)|_{ss} \not\equiv 0 \tag{3.21}$$

and consequently,

$$\max_{t} |g(t)|_{ss} = m > 0$$

Since $|g(t)|_{ss} \leq B_i$ for all t and all i, then $m \leq B_i$. The last relationship implies that $m > 0$ is a lower bound of the set $\{B_n\}$. Since m is positive, this contradicts the hypothesis of zero being the *greatest* lower bound, and the assumption in (3.21) must be false. Hence, the system is asymptotically stable.

Definition

The notation $S_{k_1}^{k_2}$ will be used to represent the input-output plane sector which is bounded by the lines $k_1 z$ and $k_2 z$ as shown in Figure 4.

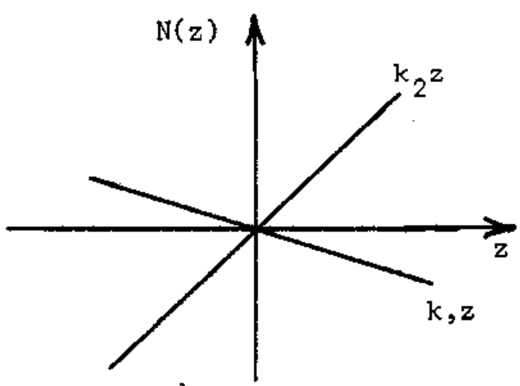

Figure 4. A Sector $S_{k_1}^{k_2}$ of the Input-Output Plane

Theorem IV

The system $R_o(N_o,H_o)$ is globally asymptotically stable if the given nonlinearity is confined to the interior of any sector of the form

$$S_{k - 1/A_k}^{k + 1/A_k}$$

where k is any number in the set I and

$$A_k = \int_o^{\infty} |h_k(t)|dt$$

<u>Proof</u>. This proof will follow the method outlined in Chapter II. Consider the representation $R_k(N_k,H_k)$ of the given system $R_o(N_o,H_o)$. Since $k \varepsilon$ I, the transient component of g(t) is stable:

$$\lim_{t\to\infty} g_{trans}(t-t_0) = \lim_{t_o\to-\infty} g_{trans}(t-t_0) = 0$$

Since the nonlinearity N_o is assumed to satisfy the conditions of Theorem I, one may start with arbitrary finite initial conditions and allow the system sufficient time to settle to its bounded steady-state motion,

$$g_{ss}(t) = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx$$

$$f_k(t) = N_k[-g_{ss}(t)] \qquad \text{for all t.}$$

Assuming that the bounded steady-state solution is not identically zero, one has

$$g_{ss}(t) \not\equiv 0$$

which implies that

$$f_k(t) \not\equiv 0 \tag{3.22}$$

for if (3.22) were not true one would have

$$g_{ss}(t) = \int_{-\infty}^{t} f_k(x)h_k(t-x) \equiv 0$$

which contradicts the assumption. Since $f_k(t)$ is bounded one may write

$$M \overset{\Delta}{=} \max_t |f_k(t)| = |f_k(t_{max})|$$

and it follows from (3.22) that M is strictly positive. One may relate the output of the linear plant $g_{ss}(t)$ to the quantity M in the following manner:

$$g_{ss}(t) = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx$$

and

$$\begin{aligned} g_{ss}(t) &\le \int_{-\infty}^{t} |f_k(x)||h_k(t-x)|dx \\ g_{ss}(t) &\le M \int_{-\infty}^{t} |h_k(t-x)|dx \end{aligned} \tag{3.23}$$

Letting y = t - x, one has

$$\int_{-\infty}^{t} |h_k(t-x)dx = \int_{0}^{\infty} |h_k(y)|dy = A_k$$

and (3.23) may be written as

$$g_{ss}(t) \leq MA_k \quad \text{for all } t. \tag{3.24}$$

By hypothesis the original nonlinearity $N_o(z)$ lies strictly within the sector

$$S_{k - 1/A_k}^{k + 1/A_k}$$

It follows that in the representation $R_k(N_k, H_k)$ under consideration, the nonlinearity $N_k(z)$ is confined to the interior of the sector, as shown in Figure 5,

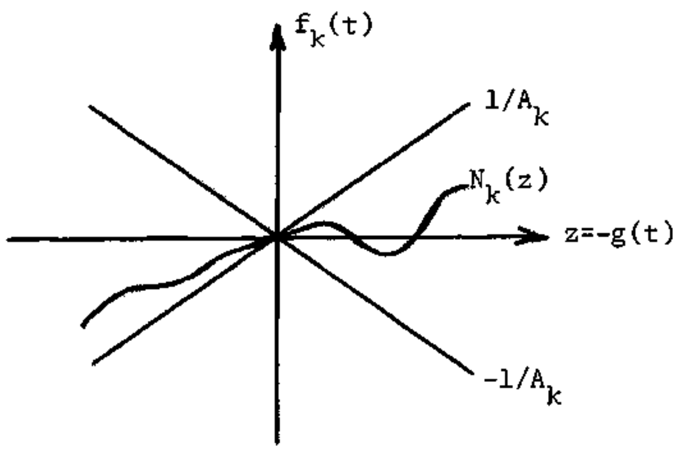

Figure 5. The Sector, $S_{-1/A_k}^{1/A_k}$

which implies that

$$\frac{|f_k(t)|}{|g(t)|} < \frac{1}{A_k} \quad \text{for } g(t) \neq 0.$$

Thus, for all instants of time for which $g(t) \neq 0$, one has

$$|g(t)| > |f_k(t)|A_k \quad \text{for all } t \text{ with } g(t) \neq 0. \tag{3.25}$$

Whereas, for those instants of time, if any, for which $g(t) = 0$,

$$|g(t)| = |f_k(t)| = 0 \quad \text{for all } t \text{ with } g(t) = 0. \tag{3.26}$$

Statements (3.25) and (3.26) give the relationship between $g(t)$ and $f_k(t)$ for all instants of time. Consider in particular the instant(s) of time t_{max} when $|f_k(t)|$ attains its maximum value

$$M = |f_k(t_{max})| > 0$$

$$g_{ss}(t_{max}) \neq 0$$

Thus, the relationship which governs $g(t)$ and $f_k(t)$ at $t = t_{max}$ is (3.25), and

$$|g_{ss}(t_{max})| > |f_k(t_{max})|\, A_k \tag{3.27}$$

$$|g_{ss}(t_{max})| > MA_k$$

The result expressed in (3.27) contradicts the relationship (3.24) which must hold for *all* t. This contradiction implies that the starting assumption, namely, $g_{ss}(t) \not\equiv 0$, is not true. Hence, the steady-state solution must be identically zero, i.e.,

$$g_{ss}(t) \equiv 0$$

and the system is asymptotically stable in the large.

Definitions

The notation $S_{+k_1}^{k_2}$ will be used to represent the semi-sector in the first and fourth quadrants (z>0) bounded by k_1z and k_2z. The notation $S_{-k_3}^{k_4}$ represents the semi-sector in the second and third quadrants (z<0) bounded by k_3z and k_4z. (See Figure 6.)

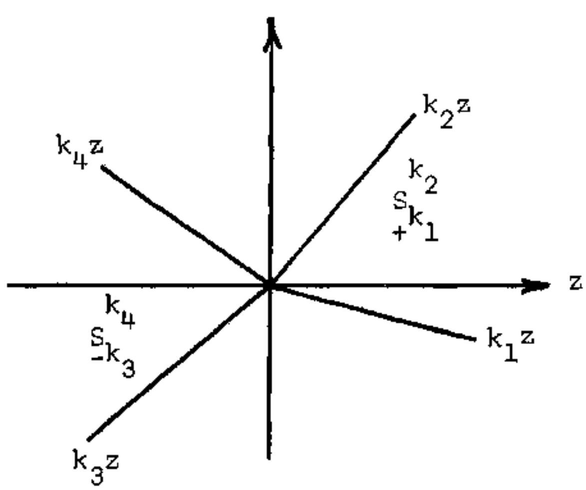

Figure 6. An Illustration of the Sectors $S_{+k_1}^{k_2}$ and $S_{-k_3}^{k_4}$

The function $h_k(t)$ can be split into a positive component and a negative component:

$$h_k(t) = h_k^+(t) + h_k^-(t)$$

where

$$h_k^+(t) = \begin{cases} h(t) & \text{whenever } h(t) \geq 0 \\ 0 & \text{whenever } h(t) < 0 \end{cases}$$

and

$$h_k^-(t) = \begin{cases} 0 & \text{whenever } h(t) \geq 0 \\ h_k(t) & \text{whenever } h(t) < 0 \end{cases}$$

Furthermore, let

$$A_k^+ = \int_0^\infty |h_k^+(t)|dt$$

and

$$A_k^- = \int_0^\infty |h_k^-(t)|dt$$

These defining relationships will be used in Theorems V(i) and V(ii) which follow.

<u>Theorem V(i)</u>

The system $R_o(N_o,H_o)$ is asymptotically stable in the large if the given nonlinearity N_o is confined over the interval $[-B_{min},B_{min}]$ to any region of the form

$$\underset{+}{S_k}^{k + 1/A_k^-} \bigcup \underset{-}{S_k}^{k - 1/A_k^+}$$

where k is any number of the set I.

<u>Proof</u>. Consider the representation $R_k(N_k,H_k)$ of the given system $R_o(N_o,H_o)$. Since k is in I, the transient component of g(t) is stable. Starting with arbitrary finite initial conditions, the system is allowed sufficient time to settle to its bounded steady-state motion. Assuming that the steady-state solution is not identically zero, one has

$$g(t) = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx \tag{3.28}$$

This implies that

$$f_k(t) \not\equiv 0 \tag{3.29}$$

where $|g(t)|$ is bounded by B_{min},

$$-B_{min} \leq g(t) \leq B_{min} \tag{3.30}$$

By hypothesis,

$$N_o(z) \subset \underset{+}{S_k}^{k + 1/A_k^-} \bigcup \underset{-}{S_k}^{k - 1/A_k^+} \qquad \text{for } -B_{min} \leq z \leq B_{min}.$$

Hence, in the presentation R_k, one has

$$N_o(z) \quad S_{+0}^{1/A_k^-} \quad S_{-0}^{-1/A_k^+} \qquad \text{for } -B_{min} \leq z \leq B_{min}.$$

A typical nonlinearity $N_k(z)$ satisfying these conditions is illustrated in Figure 7.

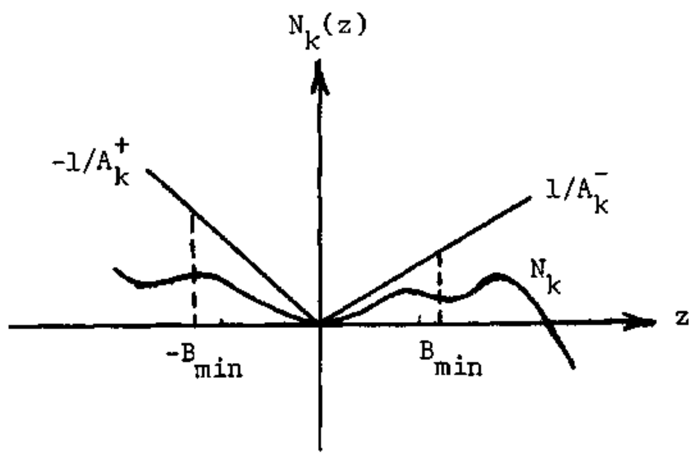

Figure 7. Location of $N_k(z)$ in the Input-Output Plane

It follows that

$$f_k(t) = N_k[-g(t)] \geq 0 \qquad -B_{min} \leq z \leq B_{min} \tag{3.31}$$

and

$$\frac{f_k(t)}{-g(t)} = \frac{|f_k(t)|}{|g(t)|} < \frac{1}{A_k^-} \qquad \text{whenever } z = -g(t) > 0 \tag{3.32}$$

$$\frac{f_k(t)}{g(t)} = \frac{|f_k(t)|}{|g(t)|} < \frac{1}{A_k^+} \qquad \text{whenever } z = -g(t) < 0 \tag{3.33}$$

$$|g(t)| = |f_k(t)| = 0 \quad \text{whenever } z = -g(t) = 0 \tag{3.34}$$

Relations (3.32), (3.33), and (3.34) may be written as

$$|g(t)| > A_k^-|f_k(t)| \quad \text{whenever } g(t) < 0 \tag{3.35}$$

$$|g(t)| > A_k^+|f_k(t)| \quad \text{whenever } g(t) > 0 \tag{3.36}$$

$$|g(t)| = |f_k(t)| = 0 \quad \text{whenever } g(t) = 0. \tag{3.37}$$

Statements (3.35), (3.36), and (3.37) give the relationships that must hold between the steady-state waveforms $|g(t)|$ and $|f_k(t)|$ for all time, i.e., for any instant of time one of the three relationships must hold depending on the sign of $g(t)$. Now, consider the expression (3.28) for the steady-state $g(t)$:

$$g(t) = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx$$

$$h_k(t) = h_k^+(t) + h_k^-(t)$$

$$h_k^+(t) \geq 0 \quad \text{for all } t \tag{3.38}$$

$$h_k^-(t) < 0 \quad \text{for all } t \tag{3.39}$$

$$g(t) = \int_{-\infty}^{t} f_k(x)h^+(t-x)dx + \int_{-\infty}^{t} f_k(x)h^-(t-x)dx \tag{3.40}$$

From Equations (3.31), (3.38), and (3.39), it follows that the first integral in Equation (3.40) is positive while the second integral is negative.

Now examine the sign of the function $g(t)$. One of three possibilities must be true:

(i) $g(t) \geq 0$ for all t.

(ii) $g(t) \leq 0$ for all t.

(iii) $g(t)$ takes positive and negative values.

Each possibility will be examined separately and a contradiction will be deduced.

(i) Suppose $g(t) \geq 0$ for all t. Since the second integral in Equation (3.40) is negative, one has

$$g(t) \leq \int_{-\infty}^{t} f_k(x)h_k^+(t-x)dx$$

and since both functions in the integrand are non-negative,

$$g(t) \leq M \int_{-\infty}^{t} h_k^+(t-x)dx$$

where

$$M = \max_t |f_k(t)|$$

$$g(t) \leq M\int_{0}^{\infty} h_k^+(y)dy$$

$$g(t) \le MA_k^+$$

$$|g(t)| \le MA_k^+ \quad \text{for all } t. \tag{3.41}$$

Consider an instant t_{max} when $|f_k(t)|$ attains its maximum value

$$M = f_k(t_{max}) \neq 0$$

and because $N_k(z)$ does not intersect the output axis except at the origin, one has

$$g(t_{max}) \neq 0$$

Therefore,

$$g(t_{max}) > 0 \tag{3.42}$$

From (3.42) it follows that the relationship that must hold between $|g(t)|$ and $|f_k(t)|$ at t_{max} is (3.36), i.e.,

$$g(t_{max}) > MA_k^+$$

but this is a contradiction to relationship (3.41), which must hold for any instant of time. Therefore, the first possibility cannot be true.

(ii) Suppose $g(t) \le 0$ for all t. Since the first integral in (3.40) is positive, one has

$$|g(t)| \le \left|\int_{-\infty}^{t} f_k(x)h_k^-(t-x)dx\right|$$

$$\le M\int^{t} |h_k^-(t-x)|dx$$

$$|g(t)| \le MA_k^- \quad \text{for all } t. \tag{3.43}$$

Consider an instant t_{max} when $|f_k(t)|$ attains its maximum value:

$$M = |f_k(t_{max}) \ne 0$$

and as before,

$$g(t_{max}) \ne 0$$

Therefore,

$$g(t_{max}) < 0 \tag{3.44}$$

Inequality (3.44) means that the relationship that holds at t_{max} is (3.35), i.e.,

$$|g(t_{max})| > A_k^-|f_k(t_{max})|$$

or

$$|g(t_{max})| > MA_k^-$$

But this is a contradiction to (3.43), which must hold for all t. Thus, the second possibility cannot be true.

(iii) Suppose that g(t) takes both positive and negative values. Consider an instant $t = t_{max}$ when $f_k(t)$ takes its maximum absolute value M, i.e.,

$$M = |f_k(t_{max})| \neq 0$$

and

$$g(t_{max}) \neq 0$$

Two cases arise regarding the sign $g(t_{max})$:

(a) Suppose $g(t_{max}) > 0$. Consider those instants of time for which $g(t) > 0$. From Equation (3.40) one obtains

$$g(t) \leq \int_{-\infty}^{t} f_k(x) h_k^+(t-x) dx$$

$$g(t) \leq MA_k^+ \qquad \text{for all } t \text{ such that } g(t) > 0$$

and since $g(t_{max}) > 0$, then

$$g(t_{max}) \leq MA_k^+ \tag{3.45}$$

But from (3.36),

$$g(t_{max}) > A_k^+ |f_k(t_{max})|$$

or

$$g(t_{max}) > MA_k^+$$

This is a contradiction to (3.45).

(b) Next, suppose $g(t_{max}) < 0$. Consider those instants of time for which $g(t) < 0$. From Equation (3.40) one has

$$|g(t)| \leq \int_{-\infty}^{t} f_k(x) h_k^-(t-x) dx$$

$$|g(t)| \leq MA_k^-$$

$$|g(t_{max})| \leq MA_k^- \tag{3.46}$$

But since $g(t_{max}) < 0$, one obtains from (3.45)

$$|g(t_{max})| > |f_k(t_{max})| A_k^-$$

$$|g(t_{max})| > MA_k^-$$

which is a contradiction to (3.46).

Thus, in every case a contradiction is obtained. Therefore, the starting assumption, namely, that $g(t) \not\equiv 0$ must be false. Hence,

ss

$g_{ss}(t) \equiv 0$, and the system is globally asymptotically stable.

Theorem V(ii)

The system $R_o(N_o,H_o)$ is asymptotically stable in the large if the given nonlinearity N_o is confined over the interval $[-B_{min},B_{min}]$ to any region of the form

$$S_{k_+}^{k - 1/A_k^-} \bigcup S_{k_-}^{k + 1/A_k^+}$$

where k is any number of the set I.

Proof. The proof is similar to the proof of the previous theorem. The main difference is that in the present case

$$f_k(t) = N_k(z) \leq 0 \qquad -B_{min} \leq z \leq B_{min}$$

whereas, in the previous theorem (see (3.31)):

$$f_k(t) = N_k(z) \geq 0 \qquad -B_{min} \leq z \leq B_{min}.$$

Remark

For those values of k for which $h_k(t) \geq 0$ at all time, one has

$$h_k^-(t) = 0 \quad \text{and} \quad A_k^- = 0, \quad \frac{1}{A_k^-} = \infty$$

and for those values of k for which $h_k(t) \leq 0$ at all time, one has

$$h_k^+(t) = 0 \quad \text{and} \quad A_k^+ = 0, \quad \frac{1}{A_k^+} = \infty$$

This means that the corresponding stability semi-sectors may extend over a full quadrant of the input-output plane.

Examples

In the following examples the same linear plant is considered with different nonlinearities. The plant has a transfer function

$$H(s) = \frac{40}{s(s+1)(s^2+0.8S+16)}$$

This plant transfer function was considered by Dewey and Jury [19] to illustrate how their results can handle nonlinearities which lie in a sector that is larger than the Popov sector. The same plant was also considered by O'Shea [37] in a recent paper in which even larger sectors were obtained. These improved results were possible at the expense of imposing on the nonlinearity additional restrictions which were not required by the Popov Theorem. In the Dewey and Jury results, the slope of the nonlinearity was restricted to lie in an interval $[-k_1, k_2]$. In the results of O'Shea the nonlinearity was further required to be monotone increasing and to have odd symmetry.

Example 1 illustrates a nonlinearity which lies in a sector larger than the Popov sector but for which the present results predict asymptotic stability in the large. In Example 2, the nonlinearity goes outside the Dewey and Jury sector and yet the system is shown to be asymptotically stable. In Example 3, the system is asymptotically stable even though a portion of the nonlinearity lies outside the Routh-Hurwitz sector.

To apply the present results to the given system, it was found convenient to simulate the linear function $H_k(s)$ on the analog computer and the functional dependence of A_k, A_k^+, and A_k^- on k was determined. The resulting graphs are shown in Figure 8.

Example 1

Consider the nonlinearity shown in Figure 9. The nonlinearity satisfies the conditions of Theorem I with $k = 0.65$, $\lambda_1 = \lambda_2 = 0$, $b_1 = b_3 = 0.6$, and $b_2 = b_4 = -0.6$. Thus,

$$M = 0.6$$

$$B_1 = MA_k = 0.6\ A_k.$$

The value of $A_{0.65}$ is obtained directly from the graph in Figure 8.

$$A_{0.65} = 2.5$$

Therefore,

$$B_1 = 0.6(2.5) = 1.5.$$

Examining the nonlinearity over the interval [-1.5,1.5], and referring to Theorem II, one can predict global asymptotic stability because the nonlinearity is within the Popov sector over the interval [-1.5,1.5].

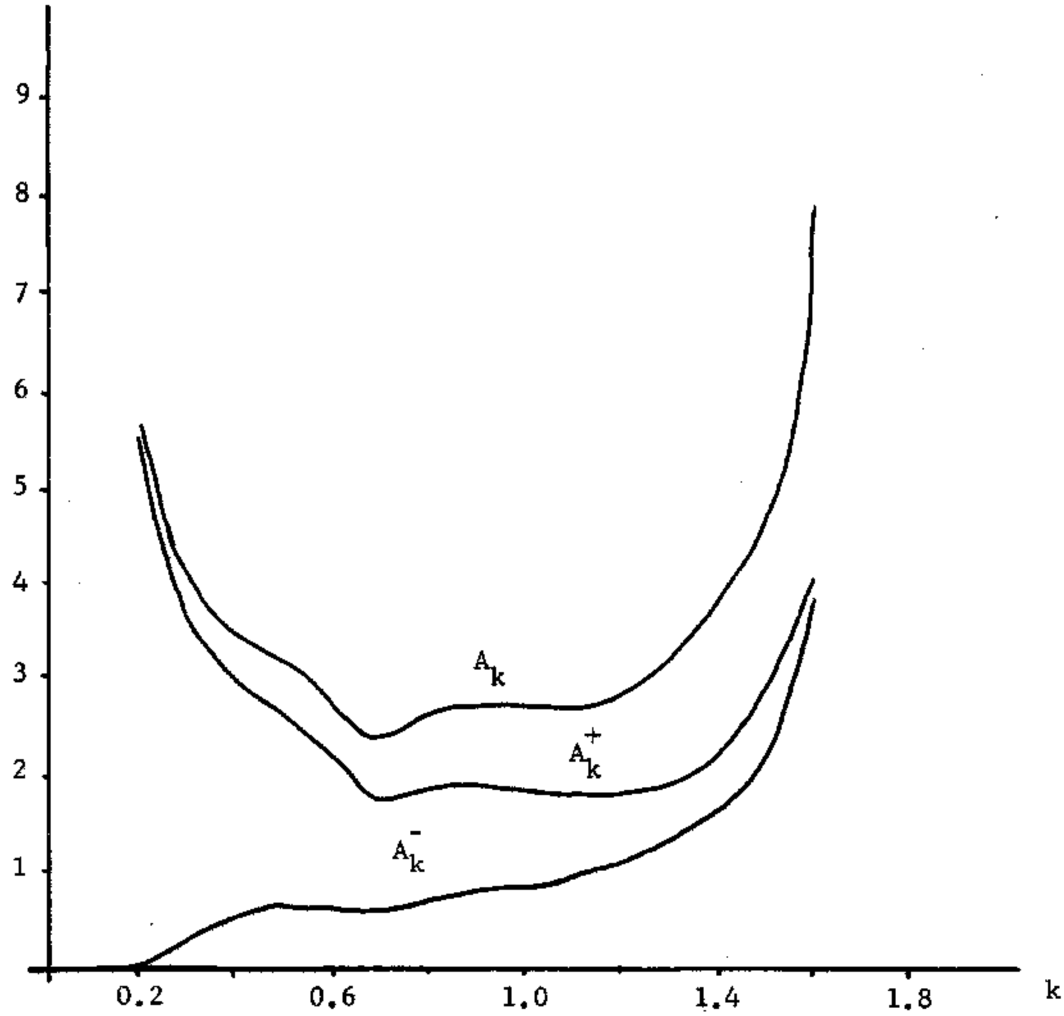

Figure 8. Variation of A_k, A_k^+, A_k^- with k

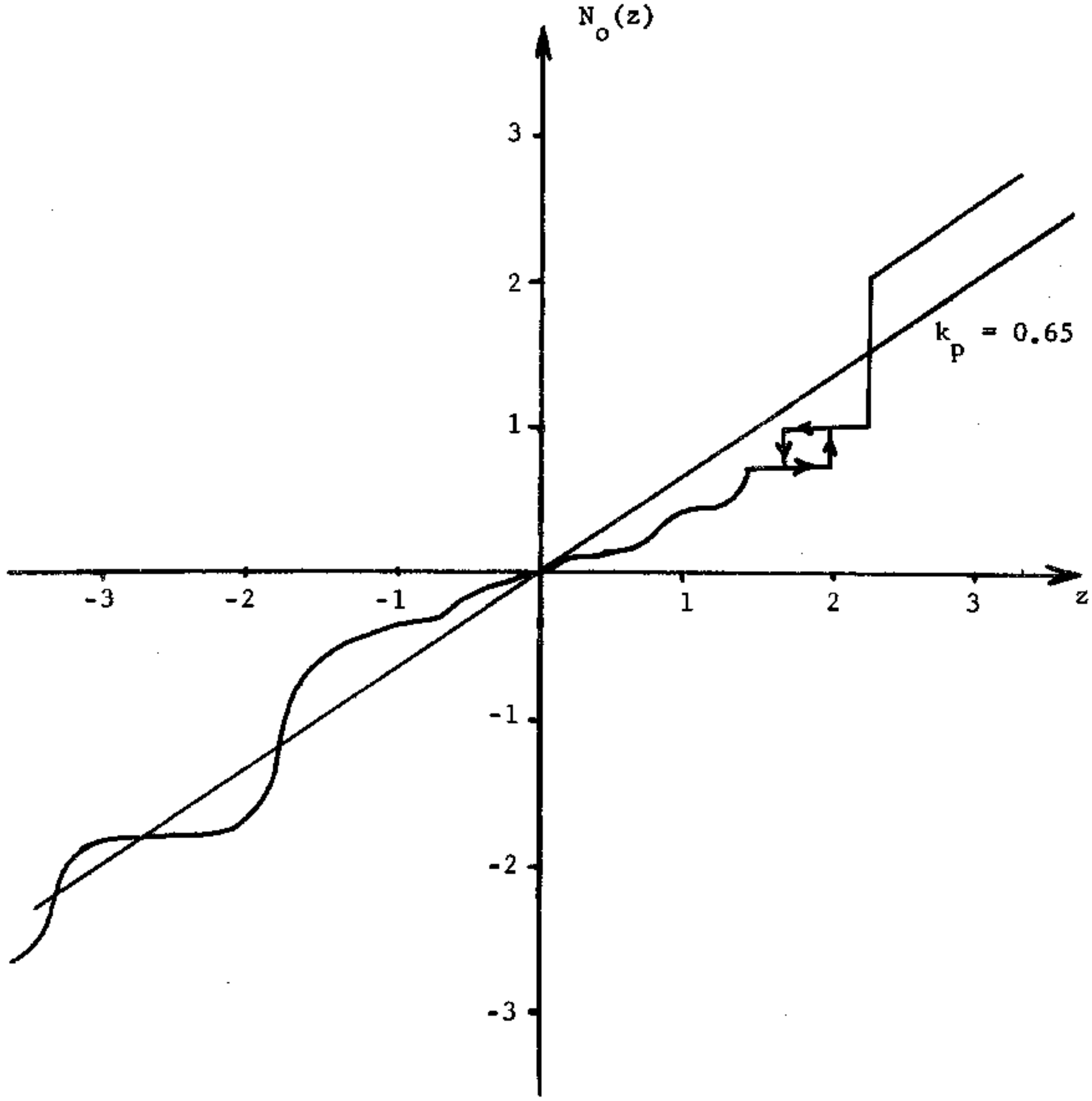

Figure 9. A Nonlinearity that Cannot Be Handled by the Popov Theorem

This system cannot be handled by the Popov criterion because $N_o(z)$ lies in a sector larger than the popov sector ($k_p = 0.65$). Furthermore, the Popov criterion does not apply because $N_o(z)$ is neither single-valued nor continuous.

Example 2

For the linear plant under consideration, the results of Dewey and Jury show that if the nonlinearity is restricted to the sector [0,1.43] and is monotone increasing with its slope restricted to the interval [0,1.43], then the system would be globally asymptotically stable.

Consider the nonlinearity in Figure 10. This nonlinearity does not satisfy the Dewey and Jury conditions since it lies in a sector larger than k = 1.43, has an infinite slope, and is not monotonically increasing. One may apply Theorem 1 with $k = 1.43$, $A_{1.43} = 4$, and $M = 0.5$. The steady-state response is bounded by

$$B_1 = 4(0.5) = 2.$$

Observing that the nonlinearity satisfies the Dewey and Jury conditions over the interval $[-B_1, B_1]$, it follows that the system is asymptotically stable in the large by Theorem II.

Example 3

In this example, the nonlinearity occupies a sector larger than the Routh-Hurwitz sector, as shown in Figure 11. One may use Theorem I with $k = 0.2$ and $M = 0.25$. From the plot of A_k the value of $A_{0.2}$ is found to be

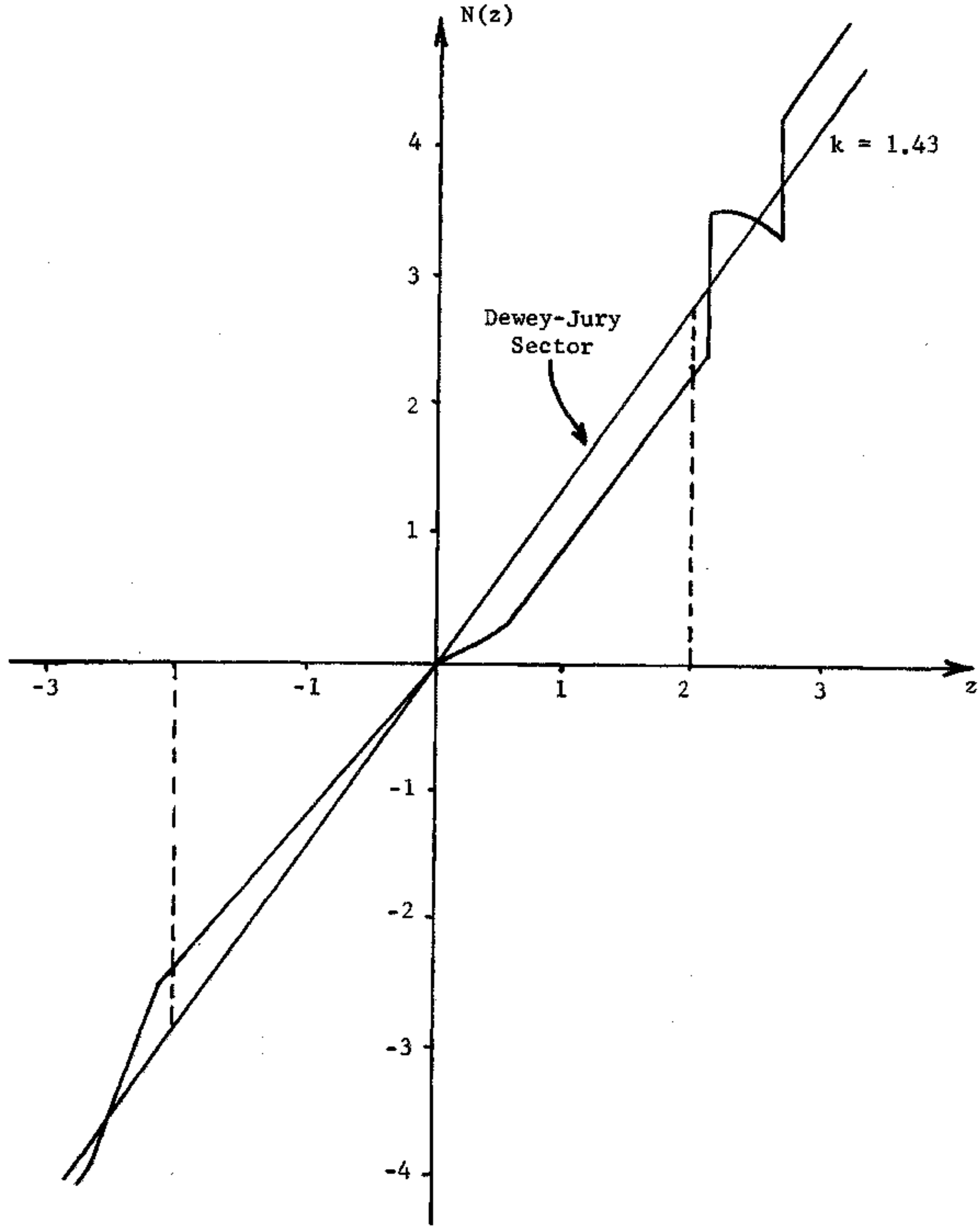

Figure 10. A Nonlinearity that Violates the Dewey-Jury Conditions

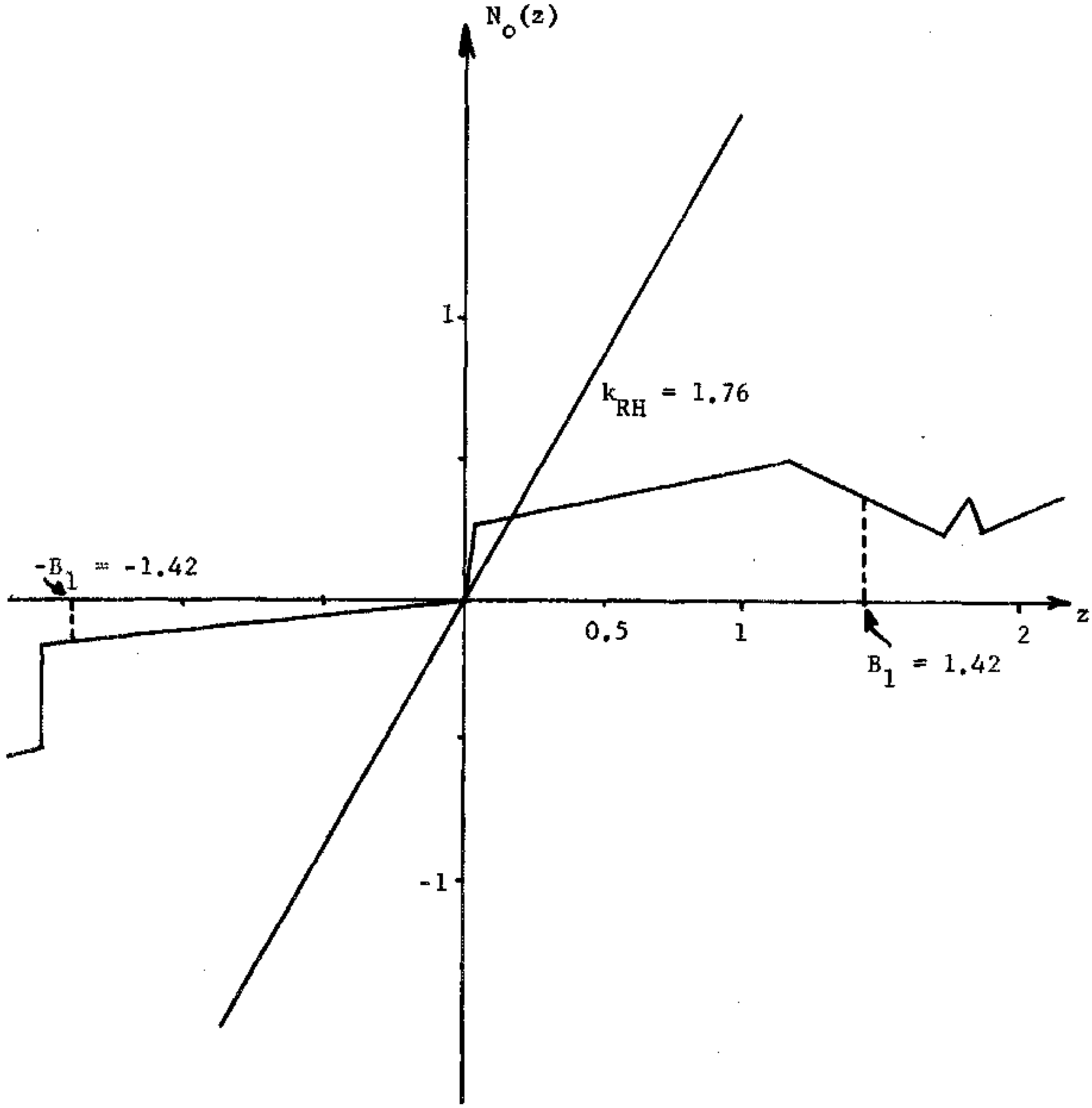

Figure 11. A Nonlinearity that Occupies a Sector Larger than k_{RH}

$$A_{0.2} = 5.71.$$

Therefore,

$$B_1 = 5.71(0.25) = 1.42.$$

The values of $A_{0.2}^{+} = 5.71$ and $A_{0.2}^{-} = 0$, are used in conjunction with Theorem V(i). For k = 0.2, the region of stability is

$$S_{+0.2}^{0.2+\infty} \bigcup S_{-0.2}^{0.2-1/5.7}$$

or

$$S_{+0.2}^{\infty} \bigcup S_{-0.2}^{0.025}$$

Since the nonlinearity lies within this stability region over the interval [-1.42,1.42], the system is asymptotically stable in the large.

In this chapter five different results have been developed for the stationary regulator system. In the next chapter corresponding results will be developed for the time-varying regulator system. The same method of approach will be used, resulting in analogous arguments and results. For this reason the detailed proofs of the theorems in the next chapter are omitted except where significant differences occur.

CHAPTER IV

TIME-VARYING SYSTEMS

This chapter deals with the stability of the time-varying regulator system. The time variation may be exhibited by either the nonlinearity or the linear plant or both. The case of the time-varying plant is considered first. The results for the time-varying nonlinearity are then presented in two separate sections. In the first section, the time-varying nonlinearity is considered to lie within a fixed region of the input-output plane, whereas in the second section the nonlinearity is considered to vary within a region which is itself time-varying. The application of the results to a system where the nonlinearity and the plant are both time-varying is illustrated by an example.

The Method Applied to the Time-Varying System

This section describes the specific application of the general method to the time-varying system. The conceptual steps of the method, as outlined in Chapter II, are still applicable. Some of the steps, however, require a treatment different from that considered for the stationary system; and it is the objective of this section to point out those considerations in the method that are peculiar to the time-varying system. This will allow the proofs of the theorems of this chapter to be presented more briefly, since the proofs will only indicate those steps which differ from the corresponding steps in the proofs which differ from the corresponding steps.

As in Chapter II, one may write the expression for g(t) as the sum of the transient and the steady state components:

$$g(t) = g(t)_{trans} + g(t)_{ss}$$

$$= g(t)_{trans} + \int_{t_o}^{t} f(x)h(t,x)dx$$

where $g(t)_{ss}$ is now represented as a superposition integral involving the impulse response h(t,x) of the linear plant. The steps of the method are modified for the time-varying system in the following manner:

(i) Examine g(t) as $t \to \infty$. It should be noted here that the behavior of g(t) as $t \to \infty$ may not be obtained by letting $t_o \to -\infty$, as was the case for the stationary system.

(ii) Verify that the transient component $g(t)_{trans}$ decreases asymptotically to zero, i.e.,

$$\lim_{t\to\infty} g(t)_{trans} = 0$$

This is not as easily verified as for the stationary system.

(iii) Allow the system sufficient time for the transient component to die out and examine the resulting waveform g(t) which is now equal to the steady-state component, i.e., as $t \to \infty$

$$g(t) \to g(t)_{ss} = \lim_{t\to\infty} \int_{t_o}^{t} f(x)h(t,x)dx$$

The steady-state waveform may possibly be an unbounded function of time.

(iv) Establish sufficient conditions for the system so that $g(t)_{ss}$ would be a bounded function. The considerations involved in the bounding of the superposition integral are different from those used in the bounding of the convolution integral of the stationary system. This is explained in detail later in the chapter.

(v) Assume that the bounded $g(t)_{ss}$ is not identically zero.

(vi) Impose certain conditions on the linear and nonlinear portions of the system, such as restricting the nonlinearity to lie in a certain region of the input-output plane.

(vii) Use the conditions in (vi) and the assumption in (v) to deduce a contradiction. This contradiction implies that if the conditions in (vi) are satisfied the assumption in (v) cannot be true. Thus, $g(t)_{ss}$ is identically zero and the system is asymptotically stable in the large. The conditions in (vi) are, therefore, sufficient conditions for stability.

The considerations involved in steps (ii) and (iv) are explained in detail in the following paragraphs.

The Stability of the Transient Response of a Time-Varying Plant

There is no known general method for the solution of homogeneous differential equations with varying coefficients. Some methods are applicable to specific classes of such equations and some iterative numerical methods may be used to obtain approximate solutions. Consequently, the question of the stability of the transient response of a time-varying plant is not as simple as in the stationary case. It was sufficient for the stationary plant to require that its poles be in the left half plane to guarantee the stability of its unforced response. For the time-varying plant it is, in general, necessary to solve directly the homogeneous linear differential equation of the specific system to verify the stability of the transient response. Furthermore, in the rest of this chapter, the results depend upon a knowledge of the plant impulse response h(t,x), which can be obtained from the differential equation governing the linear plant. The methods for obtaining h(t,x) [5] involve a knowledge of the homogeneous solution which, as mentioned above, is also needed to verify the stability of the transient response. Thus, in the remainder of this chapter it will be assumed that:

(a) The open loop linear plant is specified in terms of a linear differential equation with varying coefficients.

(b) An exact solution can be obtained for the differential equation in (a), or at least it can be predicted that its solution is asymptotically stable. If an exact solution is not available, a numerical solution may reveal whether the transient response is stable or unstable.

(c) The impulse response h(t,x) of the linear plant is given, or can be obtained from the exact solution of the homogeneous differential equation.

Bounding the Steady-State Waveform

A key step that recurred in all of the proofs of the previous chapter was the bounding of the steady state response $g_{ss}(t)$. This was effected in the following manner:

$$g(t)_{ss} = \lim_{t\to\infty} g(t) = \lim_{t\to\infty} \int_{t_o}^{t} f(x)h(t-x)dx$$

$$= \lim_{t_o\to-\infty} \int_{t_o}^{t} f(x)h(t-x)dx = \int_{-\infty}^{t} f_k(x)h_k(t-x)dx$$

Taking absolute values, one obtains

$$|g(t)_{ss}| \le \int_{-\infty}^{t} |f_k(x)||h_k(t-x)|dx \tag{4.1}$$

$$\le M\int_{-\infty}^{t} |h_k(t-x)|dx$$

$$\le MA_k$$

where $M = \max_t |f(t)|$ and $A_k = \int_0^{\infty} |h_k(t)|dt$

This step is also important in the proofs of the present chapter, but the bounding is realized in a different manner. Considering the steady-state waveform, one has

$$g(t)_{ss} = \lim_{t\to\infty} \int_{t_o}^{t} f(x)h_k(t,x)dx$$

Taking absolute values, one obtains

$$|g(t)|_{ss} \leq \lim_{t\to\infty} \int_{t_o}^{t} |f(x)|\,|h_k(t,x)|\,dx$$

$$|g(t)|_{ss} \leq \lim_{t\to\infty} M \int_{t_o}^{t} |h_k(t,x)|\,dx \tag{4.2}$$

$$\leq M \lim_{t\to\infty} \int_{t_o}^{t} |h_k(t,x)|\,dx$$

Define the function $\tilde{A}_k(t_o,t)$ as

$$\tilde{A}_k(t_o,t) \triangleq \int_{t_o}^{t} |h_k(t,x)|\,dx \tag{4.3}$$

It is assumed in all the theorems of this chapter that the function $\tilde{A}_k(t_o,t)$ is bounded for all t. With this assumption, define the quantity $\tilde{A}_k(t_o)$ as

$$\tilde{A}_k(t_o) = \begin{cases} \lim_{t\to\infty} \tilde{A}_k(t_o,t) & \text{if the limit exists} \\ \limsup_{t\to\infty} \tilde{A}_k(t_o,t) & \text{otherwise} \end{cases}$$

Also, define $\tilde{A}_k$ as

$$\tilde{A}_k \triangleq \max_{t_o} \tilde{A}_k(t_o) \tag{4.5}$$

Using the above definition, the inequality in (4.2) may be written as

$$|g(t)|_{ss} \leq M \lim_{t\to\infty} \tilde{A}_k(t_o,t) \qquad \text{if the limit exists}$$

or

$$|g(t)|_{ss} \leq M \limsup_{t\to\infty} \tilde{A}_k(t_o,t) \qquad \text{otherwise}$$

and either may be written as

$$|g(t)|_{ss} \leq M\tilde{A}_k(t_o) \tag{4.6}$$

Furthermore, using (4.5) one may write

$$|g(t)|_{ss} \leq M\tilde{A}_k \tag{4.7}$$

It should be emphasized that the bounding on the steady state waveform for the time-varying system as expressed in (4.7) has the same form as the bounding for the stationary system expressed in (4.1). Consequently, the parameter $\tilde{A}_k$ has the same role in the results of this chapter as the parameter A_k in the results of the preceding chapter. For this reason, the statements and proofs of the theorems for the time-varying system are quite similar to the corresponding theorems of the stationary system with A_k replaced by $\tilde{A}_k$. The presentation of the following sections depends upon this similarity to make the proofs briefer.

System with Time-Varying Plant

In this section the regulator system is considered to have a fixed nonlinearity and a time-varying plant. The notation H(t) is used to designate the plant whose impulse response is h(t,x). As before,

$H_k(t)$ is used to represent the linear plant in the equivalent representation R_k, and the corresponding impulse response is denoted by $h_k(t,x)$.

Theorem I.a (Lagrange Stability)

The waveforms of the given system $[N_o(z),H(t)]$ resulting from finite arbitrary conditions are bounded if there exists constants b_1, b_2, b_3, b_4, λ_1, λ_2, C, and K such that:

(i) $b_1 \geq b_2$, $b_3 \geq b_4$, $\lambda_1 \geq \lambda_2$, $C > 0$.

(ii) $b_2 + kz \leq N_o(z) \leq b_1 + kz$ for all $z > \lambda_1$

$b_4 + kz \leq N_o(z) \leq b_3 + kz$ for all $z < \lambda_1$

$|N_o(z)| \leq C$ for all $\lambda_2 \leq z \leq \lambda_1$

(iii) the open loop system $H_k(t)$ is stable, and

(iv) the function $\tilde{A}_k(t_o,t) = \int_{t_o}^{t} |h_k(t,x)|dx$ is bounded for all t.

It should be noted that the conditions in (iii) and (iv) together replace the condition $k \in I$ in Theorem I for the stationary system. The set of values of k for which hypotheses (iii) and (iv) are simultaneously satisfied is denoted by $\tilde{I}$.

Proof: As in Theorem I, the hypotheses in (i) and (ii) imply that the nonlinearity $N_k(z)$ in the representation R_k is bounded, i.e.,

$$|N_k(z)| \leq M$$

Allowing the system sufficient time, the transient component of g(t) approaches zero because $H_k(t)$ is stable. Considering the steady state component, and applying to it the bounding steps discussed earlier in the chapter, one obtains

$$|g(t)|_{ss} \leq M\tilde{A}_k(t_o,t)$$

and since $\tilde{A}_k(t_o,t)$ is bounded for all t, $g(t)_{ss}$ must be a bounded function of time.

Corollary: As indicated in (4.7), a first approximation to the bound on $|g(t)|_{ss}$ is

$$B_1 = M\tilde{A}_k$$

where $\tilde{A}_k$ is the parameter defined in (4.4) and (4.5).

Iteration of the Bound

The tightening of the bound on the steady state waveform may be performed by the same procedure described for the stationary system. Thus, the second bound B_2 is found as in (3.19), i.e.,

$$B_2 = \min_{k\varepsilon\tilde{I}} \{M_{k\cdot B_1} \cdot \tilde{A}_k\}$$

with $M_{k\cdot B_1}$ as defined in (3.16). The set $\tilde{I}$ in this case is defined as the ensemble of values of k for which hypotheses (iii) and (iv) of Theorem I.a are both satisfied. In other words, $\tilde{I}$ is the intersection of the set of k's for which $H_k(t)$ is stable and the set of k's for which $\tilde{A}_k(t_o,t)$ is bounded. The determination of these sets may not be as simple as for the stationary system.

Theorem II.a (Improvement Criterion)

The statement and proof of this theorem are identical to that of the stationary system. There are no criteria in the literature on the stability of the regulator system with a time-varying plant that are comparable to the Popov criterion. However, the improvement criterion can be used to ameliorate the effectiveness and flexibility of the criteria presented in this chapter.

Theorem III.a (Global Asymptotic Stability)

The statement and proof are the same as in the previous chapter.

Theorem IV.a (Global Asymptotic Stability)

The system $[N_o(z),H_o(t)]$ is asymptotically stable in the large if the nonlinearity is confined to the interior of any sector of the form

$$S_{k - 1/\tilde{A}_k}^{k + 1/\tilde{A}_k}$$

where k is a number in the set $\tilde{I}$.

The structure of the proof of this theorem is the same as for the stationary system with $\tilde{A}_k$ replacing A_k. The improvement criterion used with this theorem implies that the nonlinearity must be confined to the indicated sector only over the interval $[-B_{min},B_{min}]$.

Theorems V(i).a and V(ii).a (Global Asymptotic Stability)

Definitions. In the stationary system the impulse response h(t) was written as the sum of its positive and negative components $h^+(t)$ and $h^-(t)$, respectively. Because the impulse response h(t,x) of the time-varying plant is a function of two variables, the positive and negative components are identified in a different manner. In the expression for the steady state of g(t), namely,

$$g_{ss}(t) = \int_{t_o}^{t} f(x)h(t,x)dx$$

the integration is performed with respect to the x variable. Therefore, the separation of $h(t,x)$ into h^+ and h^- is performed with respect to the x variable. Thus, for any fixed t, $h^+(t,x)$ and $h^-(t,x)$ are defined as

$$h^+(t,x) \triangleq \begin{cases} h(t,x) & \text{whenever } h(t,x) \geq 0 \text{ with } -\infty < x < t \\ 0 & \text{whenever } h(t,x) < 0 \text{ with } -\infty < x < t \end{cases}$$

$$h^-(t,x) \triangleq \begin{cases} 0 & \text{whenever } h(t,x) \geq 0 \text{ with } -\infty < x < t \\ h(t,x) & \text{whenever } h(t,x) < 0 \text{ with } -\infty < x < t \end{cases}$$

The $h^+(t,x)$ and $h^-(t,x)$ are illustrated in Figure 12. For any given t, the functions $\tilde{A}^+(t_o,t)$ and $\tilde{A}^-(t_o,t)$ are defined as

$$\tilde{A}^+(t_o,t) \triangleq \int_{t_o}^{t} |h^+(t,x)|dx$$

$$\tilde{A}^-(t_o,t) \triangleq \int_{t_o}^{t} |h^-(t,x)|dx$$

where these functions are also illustrated in Figure 12. Furthermore, if $\tilde{A}^+(t_o,t)$ and $\tilde{A}^-(t_o,t)$ are bounded functions, one may define $\tilde{A}^+(t_o)$, $\tilde{A}^-(t_o)$, $\tilde{A}^+$, and $\tilde{A}^-$ as before:

$$\tilde{A}^+(t_o) \triangleq \begin{cases} \lim\limits_{t\to\infty} \tilde{A}^+(t_o,t) & \text{if the limit exists} \\ \limsup\limits_{t\to\infty} \tilde{A}^+(t_o,t) & \text{otherwise} \end{cases}$$

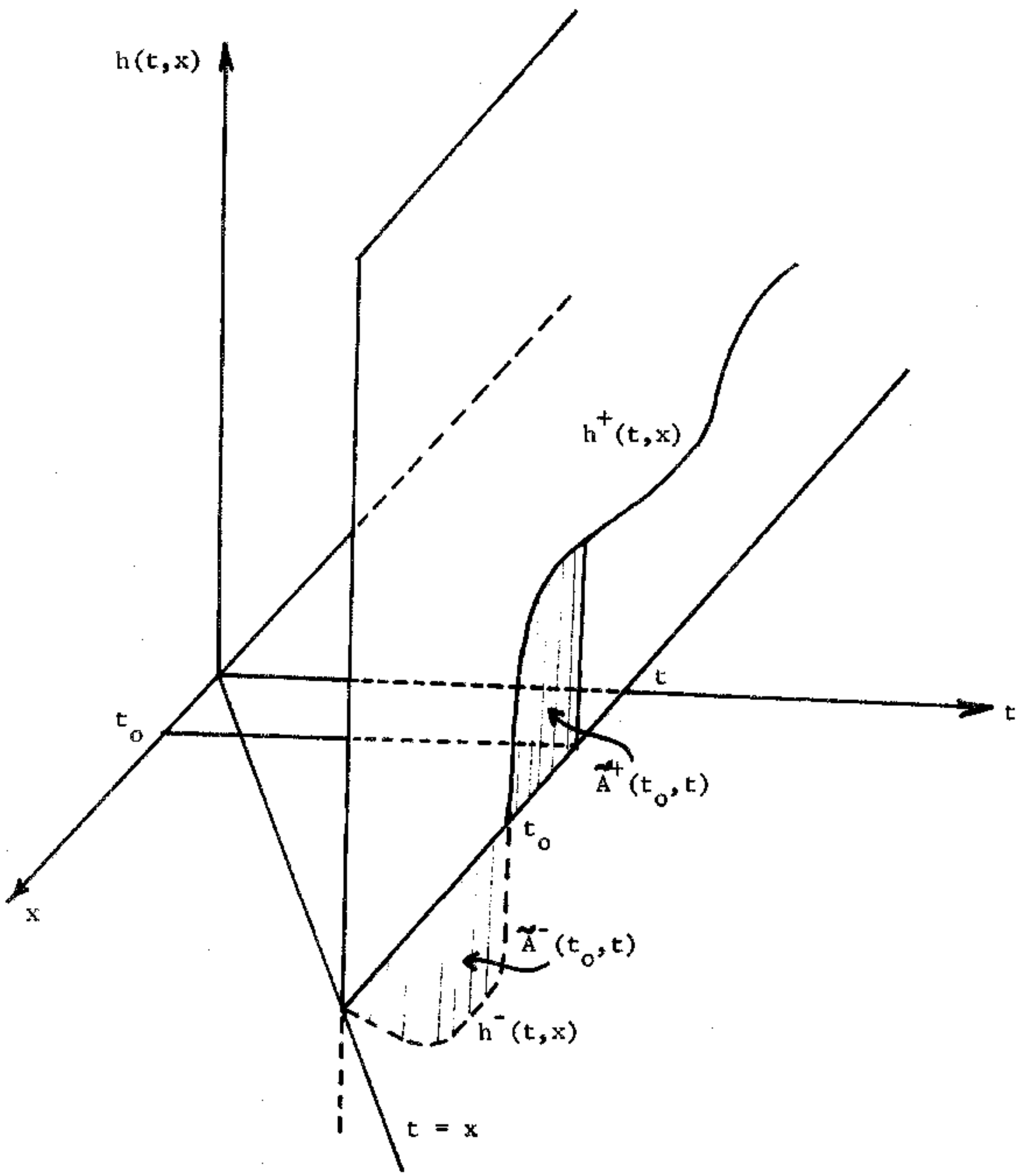

Figure 12. An Illustration Showing $h^+(t,x)$, $h^-(t,x)$ $A^+(t_o,t)$, and $A^-(t_o,t)$

$$A^-(t_o) = \begin{cases} \lim\limits_{t\to\infty} \tilde{A}^-(t_o,t) & \text{if the limit exists} \\ \limsup\limits_{t\to\infty} \tilde{A}^-(t_o,t) & \text{otherwise} \end{cases}$$

$$\tilde{A}^+ = \max_{t_o} \tilde{A}^+(t_o)$$

$$\tilde{A}^- = \max_{t_o} \tilde{A}^-(t_o)$$

For certain systems the parameters $\tilde{A}^+(t_o)$ and $\tilde{A}^-(t_o)$ are independent of t_o, and the maximization over t_o is not necessary.

Statement of the Theorems. Using these definitions, Theorems V(i).a and V(ii).a are stated in the following manner:

(i) The system $[N_o(z),H_o(t)]$ is asymptotically stable in the large if the given nonlinearity is confined to the interior of any region of the form

$$S_{k+}^{\,k + 1/\tilde{A}^-} \bigcup S_{k-}^{\,k - 1/\tilde{A}^+}$$

where k is any number of the set $\tilde{I}$.

(ii) The system $[N_o(z),H_o(t)]$ is asymptotically stable in the large if the given nonlinearity is confined to the interior of any region of the form

$$S_{k+}^{\,k - 1/\tilde{A}^-} \bigcup S_{k-}^{\,k + 1/\tilde{A}^+}$$

where k is any number of the set $\tilde{I}$.

The structure of the proofs is the same as in Theorems V(i) and V(ii) for the stationary system with A_k^+ and A_k^- replaced by $\tilde{A}_k^+$ and $\tilde{A}_k^-$, respectively.

Example

The system shown in Figure 13 has a time-varying plant governed by the following linear differential equation:

$$t^2 \frac{d^2y}{dt^2} + 4t \frac{dy}{dt} + y = f(t)$$

It will be shown that the system is asymptotically stable in the large if the nonlinearity is confined to the region shown in Figure 15, and if the system is put into operation for some $t_o > 0$. However, this is not the only region for which asymptotic stability can be verified using the results of this chapter.

First of all, one must show that the motion is stable in the Lagrange sense (bounded). Referring to Theorem I.a, one has to show that all of the hypotheses are satisfied. Each of the hypotheses (i) to (iv) is now shown to be satisfied:

(i) Let $b_1 = 2$, $b_2 = -2$, $b_3 = 2$, $b_4 = -2$

$\lambda_1 = 1$, $\lambda_2 = -1$, $C = 2$, and $K = 1$.

(ii) This portion of the hypothesis is satisfied with the various parameters having the numerical values in (i).

(iii) The linear portion $H_1(t)$ in the representation R_1 is shown in Figure 14. This linear system is governed by the differential equation

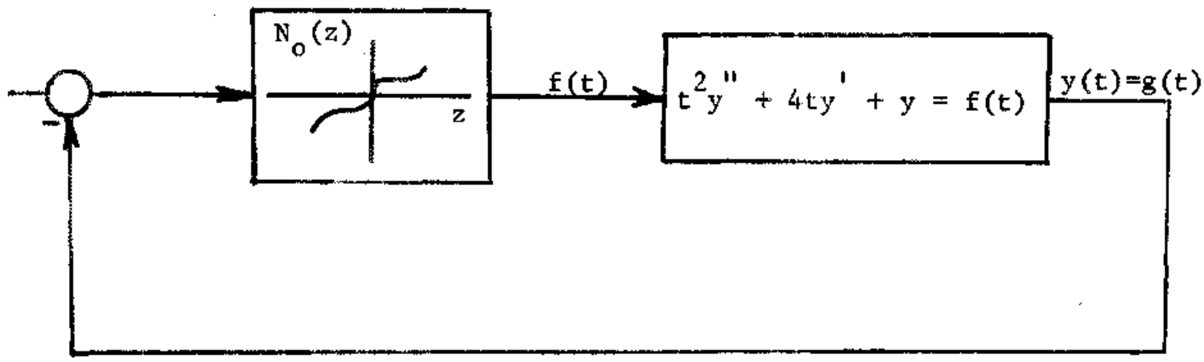

Figure 13. An Example of a Time-Varying System

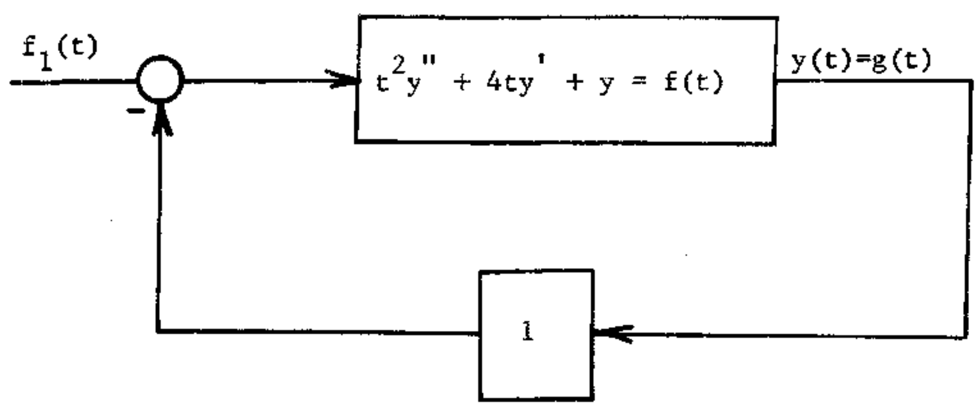

Figure 14. The Linear Block $H_1(t)$ in the Representation $R_1[N_1,H_1]$ of the System in Figure 13.

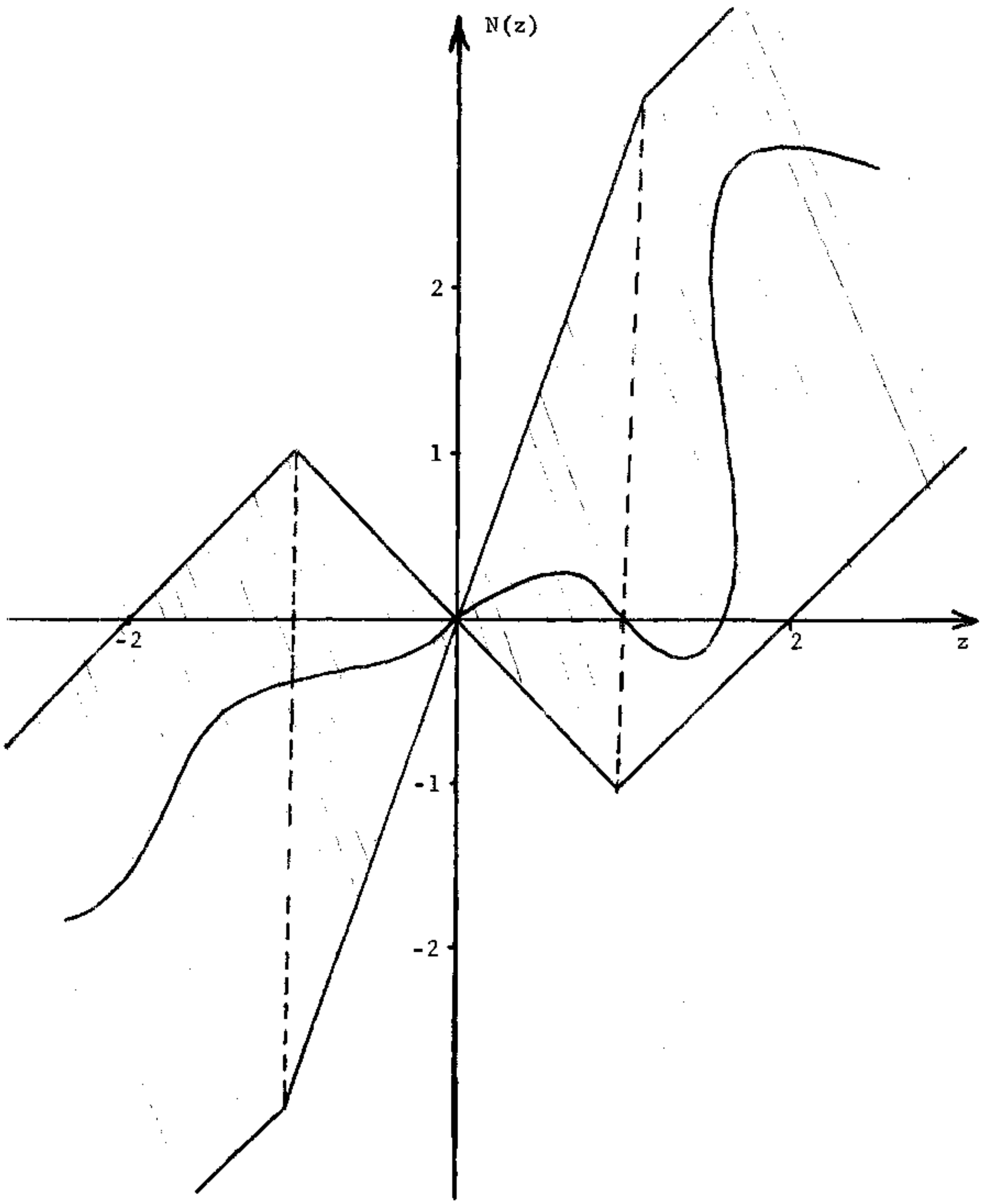

Figure 15. A Stability Region for the System in Figure 13

$$t^2y'' + 4ty' + 2y = f(t)$$

To verify the stability of the transient response one has to solve the homogeneous equation

$$t^2y'' + 4ty' + 2y = 0$$

which has the form of Euler's equation. The solution is

$$y(t)_{\text{trans}} = [g(t)_{\text{trans}}]_{k=1} = \frac{C_1}{t} + \frac{C_2}{t^2}$$

Obviously, the transient response is asymptotically stable and this portion of the hypothesis is satisfied.

(iv) The impulse response of $H_1(t)$ is [5]:

$$h_1(t,x) = \frac{t - x}{t^2} \qquad t \geq x$$

Examining the function $\tilde{A}_1(t_o,t)$,

$$\tilde{A}_1(t_o,t) = \int_{t_o}^{t} \left|\frac{t - x}{t^2}\right| dx = \frac{1}{t^2}\int_{t_o}^{t} |t-x|dx$$

and noting that for $t_o \leq x \leq t$ the quantity $(t-x)$ is nonnegative, one has

$$\tilde{A}_1(t_o,t) = \frac{1}{t^2}\int_{t_o}^{t} (t-x)dx$$

$$\tilde{A}_1(t_o,t) = \frac{1}{t^2}\left[tx - \frac{x^2}{2}\right]_{t_o}^{t}$$

$$\tilde{A}_1(t_o,t) = \frac{1}{2} - \frac{t_o}{t} + \frac{1}{2}\frac{t_o^2}{t^2}$$

Since $t \geq t_o > 0$, it follows that the function $\tilde{A}_1(t_o,t)$ is bounded for all t. Thus, the entire hypothesis of Theorem I.a is satisfied, and therefore, the steady state waveform is bounded. The corollary of the theorem gives a bound on $|g(t)|_{ss}$ as

$$|g(t)|_{ss} \leq B_1 = M_1\tilde{A}_1$$

To find $\tilde{A}_1$, one first must find $\tilde{A}_1(t_o)$,

$$\tilde{A}_1(t_o) = \lim_{t\to\infty} \tilde{A}_1(t,t_o) = \frac{1}{2}$$

$$\tilde{A}_1 = \max_{t_o} \tilde{A}_1(t_o) = \frac{1}{2}$$

Furthermore, in the representation $R_1[N_1,H_1(t)]$, one has

$$N_1(z) \leq 2$$

Thus,

$$M_1 \leq 2$$

and

$$B_1 = M_1\tilde{A}_1 = 2\left(\frac{1}{2}\right) = 1$$

$$B_1 \leq 1$$

Therefore, the steady state motion must be confined to the interval [-1,1] on the input axis of the nonlinearity.

For the given region, one observes that the nonlinearity is confined to the sector S_{-1}^{3} over the interval [-1,1]. Employing the improvement criterion and referring to Theorem IV.a, it is found that all the conditions of the latter theorem are satisfied with k = 1 and $\tilde{A}_k = \frac{1}{2}$. Therefore, the system is asymptotically stable in the large.

It should be mentioned at this point that the region of stability used in this example is only one of many possible regions having different shapes and locations that one can establish by using the results of this chapter. For instance, one may use Theorem V.a and locate a markedly different region of stability for the same linear plant.

Time-Varying Nonlinearities in Fixed Stability Regions

In this section the nonlinearity N_o is considered to be a time-varying nonlinear characteristic $N_o(z,t)$. The plant may be either stationary or time-varying. The region to which $N_o(z,t)$ is confined at all times is a fixed region of the input-output plane.

Assertion

All of the previous results developed for the stationary system and the system with time-varying plant are also valid for a system with a time-varying nonlinearity if the nonlinearity is located at all times within the stability region specified in the previous theorems. The proofs of the theorems remain the same as before. The proofs are unchanged because the time waveform f(t) in the proofs may be either the output of a stationary nonlinearity or the output of a time-varying nonlinearity.

Iteration of the Bound

Because the nonlinearity is time-varying a slight modification is needed in the process of iterating the bound on the steady state motion. Referring to Equation (3.16), the quantity $M_{k \cdot B_1}(t)$ is now a function of time, i.e.,

$$M_{k \cdot B_1}(t) = \max_{-B_1 < z < B_1} |N_k(z,t)|$$

and the quantity $M_{k \cdot B_1}$ is defined as

$$M_{k \cdot B_1} = \max_t M_{k \cdot B_1}(t)$$

Using this procedure for finding $M_{k \cdot B_1}$, all the other steps in the process of tightening the bound are identical to those for the stationary nonlinearity.

Time-Varying Nonlinearities in Time-Varying Stability Regions

In the previous results the nonlinearity $N(z,t)$ was considered to lie within a fixed region of the input-output plane for all times. In the following theorems the stability region for the time-varying nonlinearity is itself time-varying. The significance and advantage of such a consideration will be explained later in the section.

Definitions

The notation $H_{k(t)}$ is used to represent the time-varying linear system consisting of a single feedback loop with the linear block H_o in the forward path and a time-varying feedback gain $k(t)$, as shown in Figure 16. The linear block H_o may be stationary or time-variant. The impulse response function of $H_{k(t)}$ is denoted by $\underset{k(t)}{h(t,x)}$, and the functional $\underset{k(t)}{A(t_o,t)}$ is defined as

$$\underset{k(t)}{\tilde{A}(t_o,t)} = \int_{t_o}^{t} \left|\underset{k(t)}{h(t,x)}\right| dx$$

and if $\underset{k(t)}{\tilde{A}(t_o,t)}$ is bounded,

$$\underset{k(t)}{\tilde{A}(t_o)} = \begin{cases} \lim\limits_{t\to\infty} \underset{k(t)}{\tilde{A}(t_o,t)} & \text{if the limit exists} \\ \limsup\limits_{t\to\infty} \underset{k(t)}{A(t_o,t)} & \text{otherwise} \end{cases}$$

$$\tilde{A}_{k(t)} = \max_{t_o} \underset{k(t)}{\tilde{A}(t_o)}$$

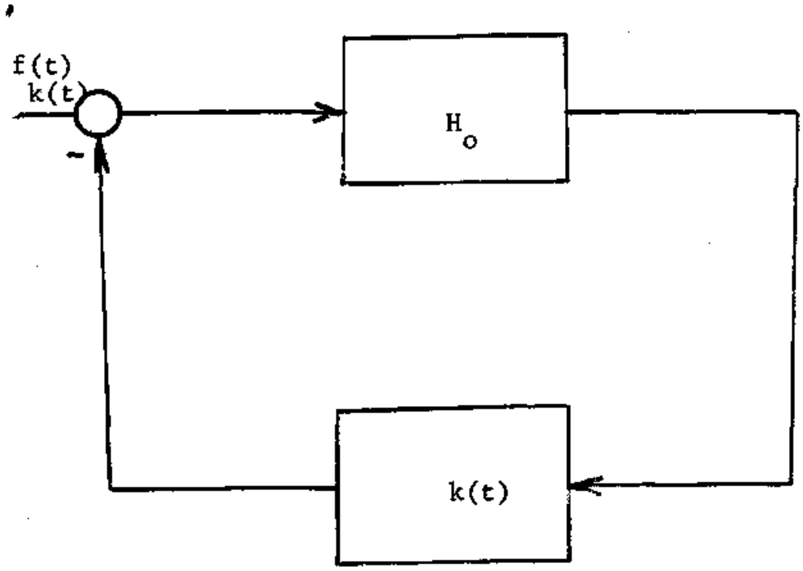

Figure 16. The System $H_{k(t)}$

The symbol $\tilde{I}[k(t)]$ is used to denote the set of all functions $k(t)$ for which the system $H_{k(t)}$ is stable and the functional $\tilde{A}_{k(t)}(t_o,t)$ is bounded. The system $H_{k(t)}$ may represent the linear block in the representation $R_{k(t)}[N_{k(t)},H_{k(t)}]$. The representation $R_{k(t)}$ is obtained by first decomposing the given nonlinearity N_o into the nonlinearity $N_{k(t)}$ and the linear time-varying gain $k(t)$, i.e.,

$$N_o(z,t) = N_{k(t)}(z,t) + k(t)z$$

and then combining $k(t)$ with H_o as explained in Chapter II.

Theorem I.b (Lagrange Stability)

The waveforms of the given system $[N_o(z,t),H_o]$ resulting from finite arbitrary initial conditions are bounded if there exist functions of time $b_1(t)$, $b_2(t)$, $b_3(t)$, $b_4(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $C(t)$, and $k(t)$ such that

(i) $b_1(t) \geq b_2(t)$, $b_3(t) \geq b_4(t)$, $\lambda_1(t) \geq \lambda_2(t)$, $C(t) > 0$.

(ii) All the functions in (i) are bounded.

(iii) $b_2(t) + k(t)z \leq N_o(z,t) \leq b_1(t) + k(t)z$ for all $z > \lambda_1(t)$.

$b_4(t) + k(t)z \leq N_o(z,t) \leq b_3(t) + k(t)z$ for all $z < \lambda_2(t)$.

$|N_o(z,t) - k(t)z| \leq C(t)$ for all $\lambda_2(t) \leq z \leq \lambda_1(t)$.

(iv) $k(t) \varepsilon \tilde{I}[k(t)]$.

Proof: Considering the representation $R_{k(t)}$, it is shown, as before, that the nonlinearity $N_{k(t)}$ is bounded, i.e.

$$|N_{k(t)}(z,t)| \leq M(t) \leq M$$

where M(t) is a bounded function defined as

$$M(t) \triangleq \max\{|b_1(t)|, |b_2(t)|, |b_3(t)|, |b_4(t)|, C(t)\}$$

and

$$M = \max_t M(t)$$

Bounding the steady state component in the usual manner, one has

$$g(t) \leq \int_{t_o}^{t} |f(x)|_{k(t)} |h(t,x)|_{k(t)} dx$$

$$\leq M\tilde{A}(t_o,t)_{k(t)}$$

Taking limits as $t\to\infty$, one obtains

$$|g(t)_{ss}| \leq M\tilde{A}_{k(t)}$$

Thus, the first bound on $|g(t)_{ss}|$ is

$$B_1 = M\tilde{A}_{k(t)}$$

The tightening of this bound is performed in the manner described for time-varying nonlinearities in the preceding section. Theorems II.b and III.b are identical to the corresponding previous theorems.

Theorem IV.b

The system $[N_o(z,t),H_o]$ is asymptotically stable in the large if the nonlinearity is confined to any time-varying sector of the form

$$S_{k(t)-1/\tilde{A}_{k(t)}}^{k(t)+1/\tilde{A}_{k(t)}}$$

where $k(t)$ is any function of the set $\tilde{I}[k(t)]$.

Proof: After considering the representation $R_{k(t)}[N_{k(t)},H_{k(t)}]$, the proof is continued as in Theorem IV.a with $\tilde{A}_{k(t)}$ replacing $\tilde{A}_k$.

Theorems V(i).b and V(ii).b

The system $[N_o(z,t),H_o]$ is asymptotically stable in the large if the nonlinearity is confined to any time-varying region of the form

$$\text{(i)} \qquad S_{+k(t)}^{k(t)+1/\tilde{A}^-_{k(t)}} \bigcup S_{-k(t)}^{k(t)=1/\tilde{A}^+_{k(t)}}$$

or of the form

$$\text{(ii)} \qquad S_{+k(t)}^{k(t)-1/\tilde{A}^-_{k(t)}} \bigcup S_{-k(t)}^{k(t)+1/\tilde{A}^+_{k(t)}}$$

where $k(t)$ is any function of the set $\tilde{I}[k(t)]$.

Proof: After considering the representation $R_{k(t)}[N_{k(t)},H_{k(t)}]$, the proof is continued as in Theorem V.a with $\tilde{A}^+_{k(t)}$ and $\tilde{A}^-_{k(t)}$ replacing A^+_k and A^-_k, respectively.

Example

This example deals with a system consisting of a time-varying plant and a time-varying nonlinearity. The system is shown in Figure 17

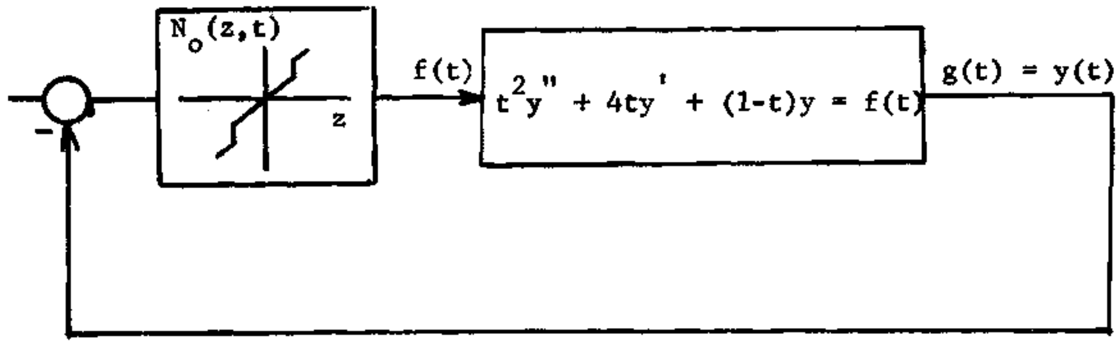

Figure 17. A System with a Time-varying Plant and a Time-varying Nonlinearity

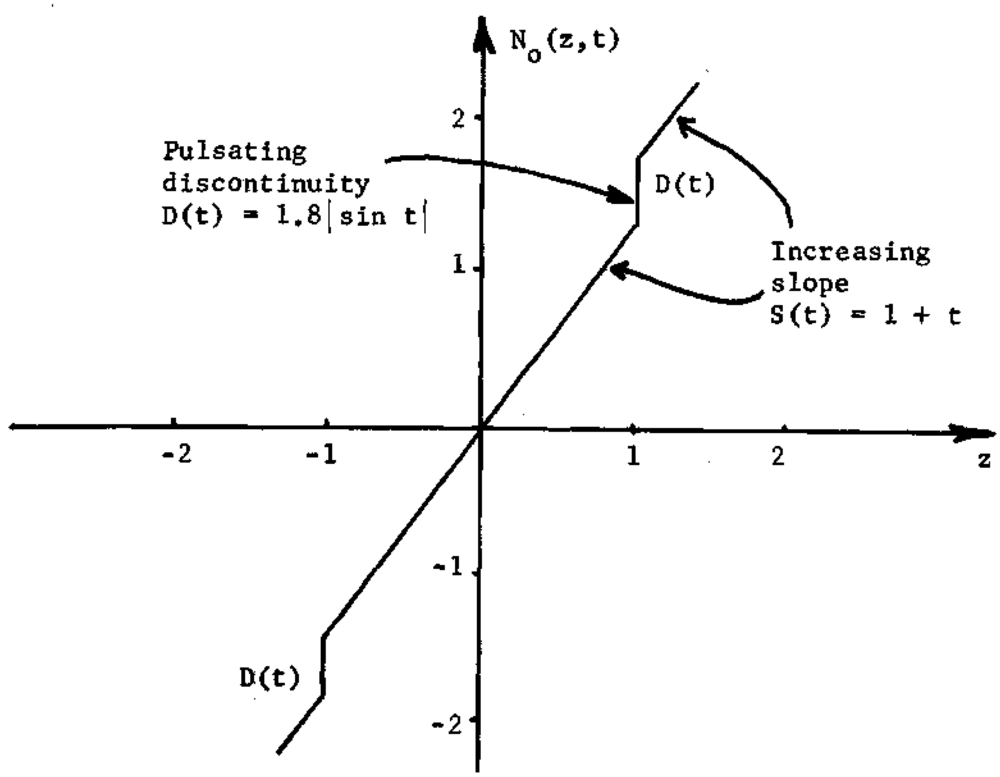

Figure 18. Details of the Nonlinearity in Figure 17

and the details of the nonlinearity are described in Figure 18. The slope of the nonlinearity increases linearly with time, i.e.,

$$S(t) = 1 + t$$

and the magnitude of the discontinuity at $z = 1$ varies sinusoidally, i.e.,

$$D(t) = 1.8|\sin t|$$

It is assumed that $t_o > 0$, and it will be shown that the system is asymptotically stable in the large.

This system may be analyzed by using the results on time-varying stability regions. All of the available results in the literature on time-varying nonlinearities are formulated in terms of a fixed stability region within which the nonlinearity is to be found at all times. If these results are to be used in analyzing the system under consideration, one would have to show that the infinite sector S_1^∞ is a sector of stability since it is the only fixed sector which contains $N_o(z,t)$ for all values of t. This is unlikely to be feasible. In general, the disadvantage in seeking a fixed stability region is that the extent of the region obtained would be governed by the extreme excursions of $N_o(z,t)$, even if these extreme excursions of the nonlinearity persist for only a short duration. In the present approach of a time-varying region, one incorporates in the analysis some information about the location of the nonlinearity at every instant of time, as well as some information on how long it stays in every location. In other words, one uses in the analysis an approximate "time record" of the behavior of the nonlinearity and not merely the information about the largest sector it occupied at some point in its history. The present approach is therefore likely to yield better results because the detailed information it utilizes is quite relevant to the stability properties of the system.

The given nonlinearity satisfies the hypothesis of Theorem I.b. This is verified by considering each part of the hypothesis:

(i) $b_1(t) = b_2(t) = 1.8|\sin t|$

$b_3(t) = b_4(t) = -1.8|\sin t|$

$\lambda_1(t) = 1 + \varepsilon, \quad \lambda_2(t) = -1 - \varepsilon$

$k(t) = 1 + t, \quad C(t) = 0.$

(ii) All of the functions in (i) are bounded, except $k(t)$ which is not required by the hypothesis to be bounded.

(iii) The nonlinearity satisfies the third part of hypothesis.

(iv) To show that $k(t) = 1 + t$ is a function of the set $\tilde{I}[k(t)]$, one has to show that the system $H_{k(t)}$ is stable and that $\tilde{A}_{k(t)}(t_o,t)$ is bounded. This may be verified as follows:

(a) The system H_{1+t} is represented in Figure 19. The governing differential equation for H_{1+t} is

$$t^2y'' + 4ty' + 2y = f(t)$$

and the solution of the unforced equation is

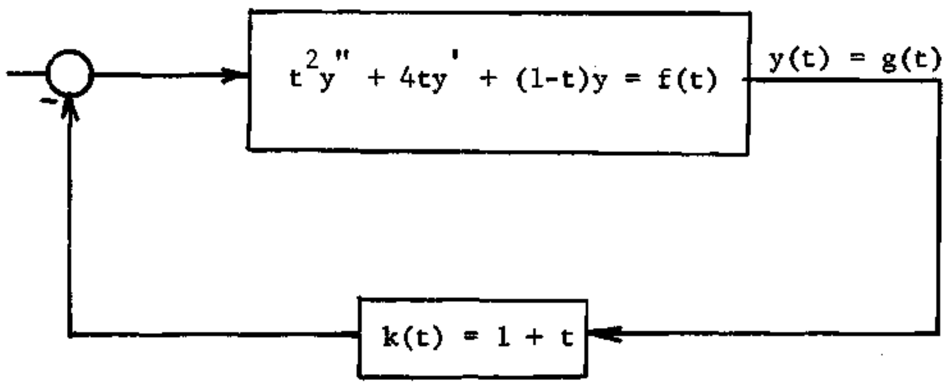

Figure 19. The System H_{1+t}

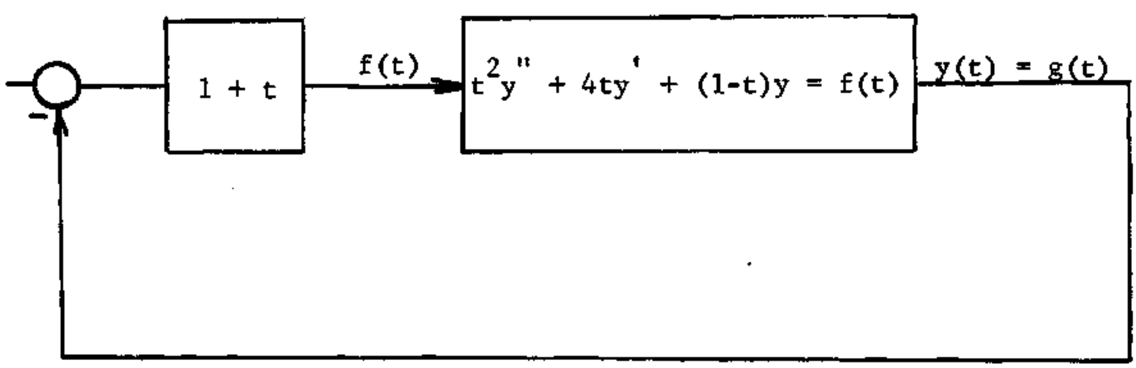

Figure 20. The Steady-State Equivalent of the System in Figure 17

$$g(t)_{trans} = y(t)_{trans} = \frac{C_1 t + C_2}{t^2}$$

where C_1 and C_2 are constants depending on the initial conditions. Since

$$\lim_{t\to\infty} g(t)_{trans} = 0$$

the system H_{1+t} is stable.

(b) The impulse response of H_{1+t} is

$$h(t,x)_{1+t} = \frac{t - x}{t^2} \qquad t \geq x$$

and

$$\tilde{A}_{1+t} = \int_{t_o}^{t} \left|\frac{t-x}{t^2}\right| dx$$

$$= \frac{1}{2} - \frac{t_o}{t} + \frac{1}{2}\frac{t_o^2}{t^2}$$

Since it is assumed that the system begins its operation for some $t_o > 0$, $\tilde{A}_{1+t}(t_o,t)$ is bounded for all $t \geq t_o$. This completes the verification of hypothesis (iv).

Thus, the system is stable in the Lagrange sense. The expression for the first bound B_1 is

$$B_1 = M\tilde{A}_{1+t}$$

where

$$M = \max_{t} M(t) = \max_{t} 1.8|\sin t| = 1.8$$

and

$$\tilde{A}_{1+t} = \max_{t_o} \tilde{A}_{1+t}(t_o)$$

with

$$\tilde{A}_{1+t}(t_o) = \lim_{t\to\infty} \tilde{A}_{1+t}(t_o,t) = \frac{1}{2}$$

Thus,

$$B_1 = 1.8\left(\frac{1}{2}\right) = 0.9$$

and

$$|g(t)|_{ss} \leq 0.9$$

After finding B_1, one may apply the improvement criterion. It is observed that the nonlinearity over the interval [-0.9,0.9] is, in effect, a linear time-varying gain $S(t) = 1 + t$. Thus, in the steady state, the behavior of the given system may be represented by the block diagram of Figure 20. But the system in Figure 20 is a linear system governed by the equation

$$t^2y'' + 4ty' + 2y = 0$$

whose solution is

$$y(t) = \frac{k_1}{t^2} + \frac{k_2}{t}$$

This system is asymptotically stable in the large, and consequently the original system is asymptotically stable in the large.

In this chapter the method was applied to the time-varying system in which the plant, or the nonlinearity, or both could be time-varying. For the purpose of simplifying the presentation, the cases of time-varying plant and time-varying nonlinearity were handled separately. The combined results, however, are immediately applicable to systems which exhibit time variation in both parts, as demonstrated in the last example. The same example also illustrated the effectiveness of the results on time-varying stability regions. In the following chapter the results are extended to sampled-data systems.

CHAPTER V

SAMPLED-DATA SYSTEMS

In this chapter the results of the previous chapters are extended to sampled-data systems. The basic system considered is the conventional feedback loop with an ideal sampler at the input of the linear plant as shown in Figure 21. The sampler is assumed to have a uniform sampling period of T seconds. The linear block H_o can be either stationary or time-varying with an impulse response h(t) or h(t,x), respectively. The nonlinearity can also be stationary or time-variant. The same method of approach can be used for the sampled-data system with some modification on the bounding of the steady state component of g(t). This modification is needed because the expression for the forced component of g(t) now appears as a discrete summation rather than in the form of an integral.

Bounding the Forced Component of g(t)

In the following derivation it is assumed that the linear block H_o is a stationary system with an impulse response h(t). Expressing the output g(t) as the sum of its transient and forced components, one has

$$g(t) = g(t)_{trans} + \sum_{n_o}^{nT=T} f(nT)h(t-nT) \qquad (5.1)$$

Since the system is stationary, the steady state motion may be obtained

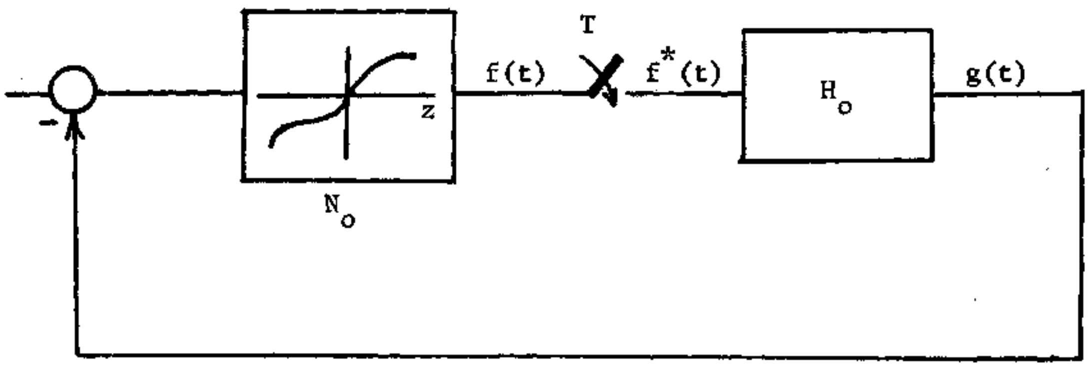

Figure 21. The Sampled-Data System under Consideration

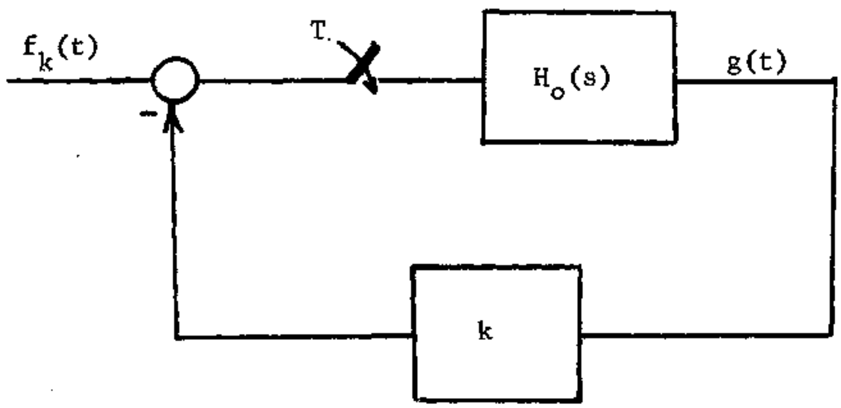

Figure 22. The Linear Block $H_k(s,T)$ in the Representation $R_k[N_k,H_k]$ for a Stationary Sampled-Data System

by letting $n_o \to -\infty$. Assuming that the stability of the transient response has been verified, one may write

$$g_{ss}(t) = \sum_{-\infty}^{nT=1} f(nt)h(t-nT) \qquad (5.2)$$

Taking absolute values on both sides of (5.2), one obtains

$$|g_{ss}(t)| \leq \sum_{-\infty}^{nT=t} |f(nT)|\,|h(t-nT)| \qquad (5.3)$$

Assuming that the boundedness of f(t) has already been established, i.e.,

$$|f(nT)| \leq M$$

the inequality in (5.3) may be further written as

$$|g_{ss}(t)| \leq M \sum_{-\infty}^{nT=t} |h(t-nT)| \qquad (5.4)$$

The summation in (5.4) is an infinite series whose terms are the uniform samples of the impulse response function. This series, if convergent, is denoted by the symbol A(T). Thus

$$A(T) \triangleq \sum_{-\infty}^{nT=t} |h(t-nT)| = \sum_{0}^{\infty} |h(nT)| \qquad (5.5)$$

Using this notation, (5.4) may be written as

$$|g_{ss}(t)| \leq MA(T) \qquad (5.6)$$

Furthermore, if one is considering a representation R_k, then

$$|g(t)|_{ss} \leq MA_k(t) \tag{5.7}$$

where

$$A_k(T) = \sum_{n=o}^{\infty} |h_k(nT)| \tag{5.8}$$

The parameter A(T) represents the absolute sum of the weighted sequence of the impulse response h(t). The samples of the weighted sequence represent the inverse z-transform of the pulsed transfer function of the sampler and H(s) in tandem.

If the linear block H_o is a time-varying system, then the output g(t) is written as

$$g(t) = g_{trans}(t-t_0) + \sum_{n_o}^{nT=t} f(nT)h(t,nT)$$

where h(t,x) is the impulse response of H_o. In this case the bound on the steady state g(t) would be written as

$$|g(t)|_{ss} \leq M\tilde{A}(T) \tag{5.9}$$

with

$$\tilde{A}(T,t,n_o) \triangleq \sum_{n_o}^{nT=t} |h(t,nT)|$$

$$\tilde{A}(t,n_o) = \begin{cases} \lim\limits_{t\to\infty} \tilde{A}(T,t,n_o) & \text{if the limit exists} \\ \limsup\limits_{t\to\infty} \tilde{A}(T,t,n_o) & \text{otherwise} \end{cases}$$

and

$$\tilde{A}(T) = \max_{n_o} \tilde{A}(T,n_o)$$

where the maximization over n_o is to be performed over those values of n_o for which the system is in operation.

Extending the Results to the Sampled-Data System

Referring to the inequalities in (5.7) and (5.9), it is noted that the bounding of the steady state component has the same form as in the continuous system with a stationary plant or a time-varying plant. Consequently, one has the following assertion.

Assertion

All the results of the previous chapters are still applicable to the sampled-data system if

(i) $A_k(T)$ replaces A_k of the stationary continuous system.

(ii) $\tilde{A}_k(T)$ replaces $\tilde{A}_k$ of the time-varying system with fixed stability regions.

(iii) $\tilde{A}_k(T)$ replaces $\tilde{A}_{k(t)}$ of the time-varying system with time-varying stability regions.

The theorems for the sampled-data system are referred to as Theorems I.c, II.c, III.c, IV.c, and V.c.

Remarks

The following considerations are peculiar to the sampled-data system.

(1) The parameters $A_k(T)$, $\tilde{A}_k(T)$, and $\tilde{A}_{k(t)}(T)$ are all functions of T. Thus, the stability properties of the system are, as should be expected, dependent on the sampling rate.

(2) The linear block $H_k(s,T)$ in the representation R_k of the original system is the feedback loop shown in Figure 22. In this case, the sum of the weighted sequence, namely, $A_k(T)$, may be determined by first finding the pulsed transfer function of $H_k(S,T)$, i.e.,

$$H_k(z) = \frac{H_o(z)}{1 + kH_o(z)}$$

The inverse z-transform of $H_k(z)$ gives the samples $h_k(nT)$, and

$$A_k(T) = \sum_{n=o}^{\infty} |h_k(nT)|$$

(3) In those cases where the sum of the infinite series indicated in (5.5) cannot be determined in closed form, and therefore must be evaluated by summing a sufficient number of terms, the question of the convergence of the series (5.5) has to be resolved first. In such cases the integral tests for the convergence of an infinite series may be conveniently used, namely, if $\int_o^{\infty} |h(t)|dt$ converges then

$$\sum_{n=o}^{\infty} |h(nT)|$$

converges.

Example

This example deals with the system shown in Figure 23. It will be verified that the system is asymptotically stable in the large if the nonlinearity N_o is confined to the region shown in Figure 15 for the first example in Chapter IV. It was shown in that example that the nonlinearity satisfies the conditions of Theorem I.a, which is referred to as Theorem I.c for the discrete system. It was found that

$$k = 1, \qquad M = 2$$

In the present example, one must find $A_1(T)$. Considering the representation R_1, the linear block $H_1(z)$ is shown in Figure 24. The pulsed transfer function is

$$H_1(z) = \frac{z - z^2}{z^2 + z + 2e^{-1} + (z-z^2)}$$

$$= \frac{z - z^2}{2(z-e^{-1})} = \frac{1}{2}\left[\frac{z}{z - e^{-1}} - \frac{z^2}{z - e^{-1}}\right]$$

The inverse transform gives

$$h_1(nT) = h_1(n) = \frac{1}{2}\left(e^{-n} - e^{-(n+1)}\right)$$

Therefore,

$$A_1(T) = \frac{1}{2}\sum_{0}^{\infty} \left|e^{-n} - e^{-(n+1)}\right|$$

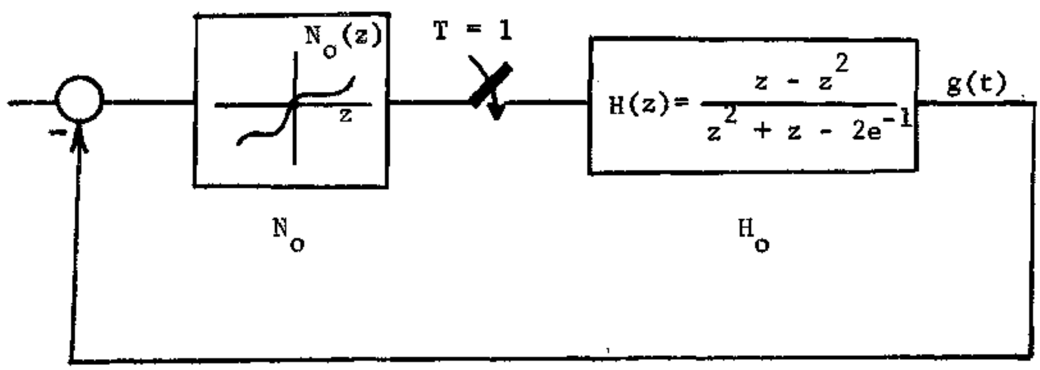

Figure 23. An Example of a Sampled-Data System

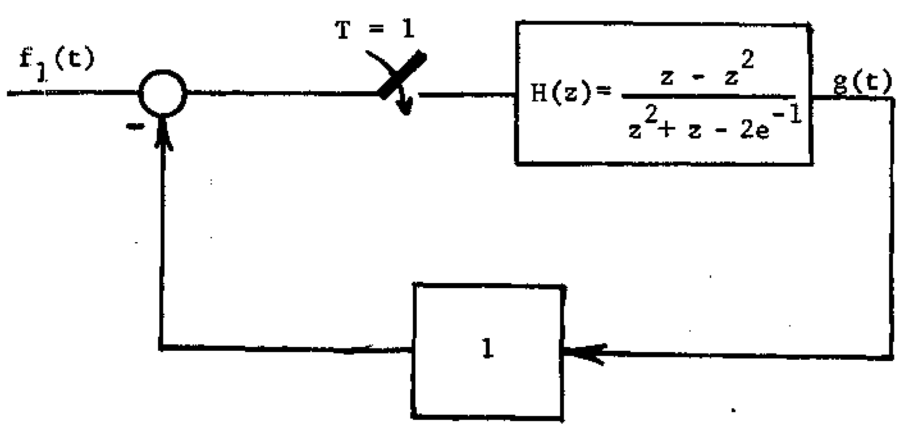

Figure 24. The Linear Plant $H_1(z)$ in the Representation R_1 of the System in Figure 23.

and since $e^{-n} > e^{-(n+1)}$ for all $n \geq 0$, the absolute value signs may be dropped. Thus,

$$A_1(T) = \frac{1}{2} \sum_{0}^{\infty} \left(e^{-n} - e^{-(n+1)}\right)$$

$$= \frac{1}{2} [e^{0} - e^{-1} + e^{-1} - e^{-2} + e^{-2} - e^{-3} + e^{-3} \cdots]$$

$$= \frac{1}{2} e^{0} = \frac{1}{2}$$

The steady-state waveform bound is

$$B_1 = MA_1(T) = 2\left(\frac{1}{2}\right) = 1$$

Thus,

$$|g(t)|_{ss} \leq 1$$

Using the improvement criterion together with Theorem IV.c, it observed that the nonlinearity is confined over the interval [-1,1] to the sector

$$S_{-1}^{3} = S_{-k}^{k + 1/A_k(T)}$$

where $k = 1$ and $A_k(T) = A_1(T) = \frac{1}{2}$. Therefore, the system is asymptotically stable in the large.

In this chapter it was shown that all of the previous results are applicable to the discrete system if the parameters A_k, $\tilde{A}_k$, and

$\tilde{A}_{k(t)}$ are replaced by their discrete counterparts $A_k(T)$, $\tilde{A}_k(T)$, and $\tilde{A}_{k(t)}(T)$, respectively. In the following chapter, the method is adapted to systems that have several nonlinear characteristics.

CHAPTER VI

SYSTEMS WITH MULTIPLE NONLINEARITIES

In this chapter the method is extended to systems having more than one nonlinear characteristic. Specifically, stability results are derived for the system shown in Figure 25, which consists of a cascade of nonlinearities alternating with linear blocks. Some of the results obtained are immediately applicable to other system configurations where the nonlinearities may occupy any positions in the overall system. In the derivation of the results, it is assumed that the system is stationary and continuous. However, these results are also applicable to the time-varying and discrete systems if the A-parameters are replaced by the $\tilde{A}$-parameters and the A(T)-parameters, respectively. Two distinct theorems on Lagrange stability are presented. The first is a generalization of Theorems I of the previous chapters, whereas the second is peculiar to the specific system under consideration. In the two preceding chapters it was possible to shorten the presentation by pointing out the particular aspects of the analysis which differed from those in the chapter on stationary systems. The material of this chapter is not easily presentable in that manner, and therefore the presentation reverts to the pattern of Chapter III. Because of the large number of variables in the system under consideration, it is convenient to use vector-matrix notation. The notations and the various definitions used are given in the following section.

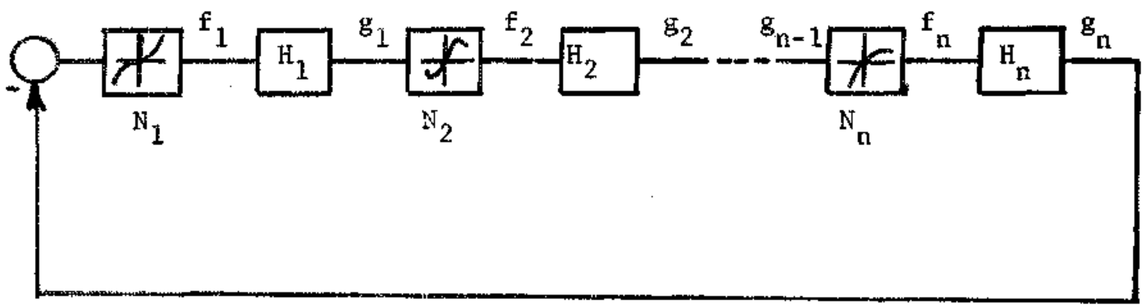

Figure 25. The Cascade System with Multiple Nonlinearities

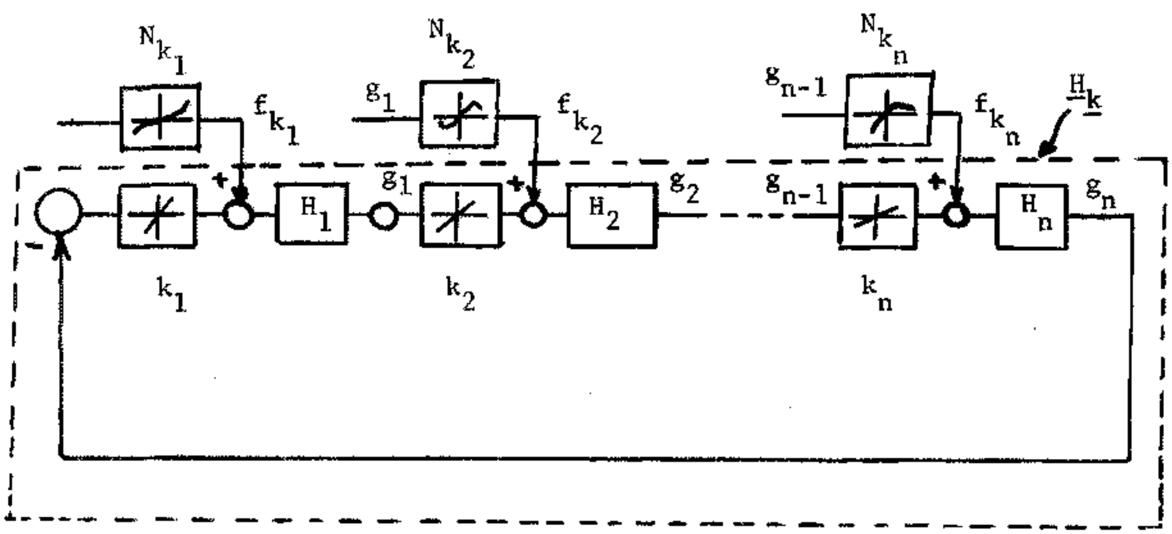

Figure 26. The Representation $R_{\underline{k}}$ and the System $\underline{H}_{\underline{k}}$

Definitions and Notations

A vector quantity will be designated by placing a bar under the symbol, e.g.

$$\underline{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \cdot \\ \cdot \\ \cdot \\ g_n(t) \end{bmatrix}$$

A square matrix will be designated by bracketing the identifying symbol, e.g.

$$[A] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix}$$

The original system in Figure 25 is said to be in the representation $R_o(\underline{N}_o, \underline{H}_o)$, where $\underline{N}_o$ and $\underline{H}_o$ are the vectors whose components are the individual nonlinearities and linear blocks, respectively. When each nonlinearity $N_i(z)$ is split into a parallel combination of a linear gain k_i and a nonlinearity N_{k_i}, as shown in Figure 26, the system is said to be in the representation $R_k(\underline{N}_k, \underline{H}_k)$, where

$$\underline{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

The linear system $\underline{H}_{\underline{k}}$ (enclosed by the dotted rectangle in Figure 26) may be regarded as having n inputs and n outputs. The inputs are the output waveform $f_{k_i}(t)$ of the nonlinearities N_i, and the outputs are the waveforms $g_i(t)$ representing the outputs of the linear blocks. Thus, the input vector is

$$\underline{f}_{\underline{k}} = \begin{bmatrix} f_{k_1} \\ f_{k_2} \\ \vdots \\ f_{k_n} \end{bmatrix}$$

and the output vector is

$$\underline{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Define k and H(s) as

$$k = k_1 k_2 \cdots k_n = \prod_{1}^{n} k_i \tag{6.1}$$

$$H(s) = H_1(s)\, H_2(s) \cdots H_n(s) = \prod_{1}^{n} H_1(s) \tag{6.2}$$

The transfer function between the input f_{kj} and the output g_i is denoted by $H_{\underline{k}\cdot ij}(s)$,

$$H_{\underline{k}\cdot ij}(s) = \begin{cases} \dfrac{(k_{j+1}k_{j+2}\cdots k_i)(H_j H_{j+1}\cdots H_i)}{1 + kH(s)} & \text{if } i \geq j \\[2ex] \dfrac{(k_{j+1}k_{j+2}\cdots k_n)(k_1 k_2 \cdots k_n)(H_j \cdots H_n)(H_1 \cdots H_i)}{1 + kH(s)} & \text{if } i < j \end{cases} \tag{6.3}$$

and the corresponding impulse response is

$$h_{\underline{k}\cdot ij}(t) \longleftrightarrow H_{\underline{k}\cdot ij}(s)$$

and

$$A_{\underline{k}\cdot ij} = \int_0^\infty \left| h_{\underline{k}\cdot ij}(t) \right| dt \tag{6.4}$$

The corresponding matrices are $[h_{\underline{k}}(t)]$ and $[A_{\underline{k}}]$.
The set $\underline{I}$ is the set of all vectors $\underline{k}$ for which the zeros of $1 + kH(s)$ are in the left half plane with k as defined in (6.1). Thus, if $\underline{k} \in \underline{I}$, then all of the transfer functions $H_{\underline{k}\cdot ij}(s)$ in (6.3) are stable and consequently, the parameters $A_{\underline{k}\cdot ij}$ in (6.4) are finite numbers.

Lagrange Stability

Theorem I.d

The waveforms in the system of Figure 25 resulting from finite arbitrary initial conditions are bounded if there exist vectors $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, $\underline{b}_4$, $\underline{\lambda}_1$, $\underline{\lambda}_2$, $\underline{C}$, and $\underline{k}$ such that

(i) $\underline{b}_1 \geq \underline{b}_2$, $\underline{b}_3 \geq \underline{b}_4$, $\underline{\lambda}_1 \geq \underline{\lambda}_2$, $\underline{C} \geq 0$

(ii) $\underline{b}_2 + \underline{k}^T\underline{z} \leq \underline{N}_o \leq \underline{b}_1 + \underline{k}^T\underline{z}$ for all $\underline{z} \geq \underline{\lambda}_1$

$\underline{b}_4 + \underline{k}^T\underline{z} \leq \underline{N}_o \leq \underline{b}_3 + \underline{k}^T\underline{z}$ for all $z \leq \underline{\lambda}_2$

$|N_{\underline{o}i}| \leq C_i$ for all $\lambda_{2i} \leq z_i \leq \lambda_{1i}$

(iii) $\underline{k} \in \underline{I}$.

where all equalities and inequalities involving vectors are understood to be relations between corresponding components of the vectors.

Proof: Considering the output vector $\underline{g}(t)$ in the representation $R_{\underline{k}}$ which may be written as

$$\underline{g}(t) = \underline{g}_{trans}(t-t_0) + \int_{t_o}^{t} [h_{\underline{k}}(t-x)]\underline{f}_{\underline{k}}(x)dx \qquad (6.5)$$

Since $\underline{k} \in \underline{I}$ the transient components are stable, i.e. $\lim_{t\to\infty} \underline{g}_{tran}(t) = \underline{0}$, in the steady state (6.5) may be written as

$$\underline{g}_{ss}(t) = \lim_{t\to\infty} \int_{t_o}^{t} [h_{\underline{k}}(t-x)]\underline{f}_{\underline{k}}(x)dx \qquad (6.6)$$

$$= \int_{-\infty}^{t} [h_{\underline{k}}(t-x)]\underline{f}_{\underline{k}}(x)dx$$

The ith component $\underline{g}(t)_{ss}$ may be written as

$$g(t)_{ss\cdot i} = \int_{-\infty}^{t} (h(t-x)_{\underline{k}\cdot i1} f_{k_1}(x) + h(t-x)_{\underline{k}\cdot i2} f_{k_2}(x) + \cdots h(t-x)_{\underline{k}\cdot in} f_{k_n}(x))dx \qquad (6.7)$$

Taking absolute values in (6.7), one has

$$|g(t)_{ss\cdot i}| \le \int_{-\infty}^{t} |h(t-x)_{\underline{k}\cdot i1}||f(x)_{k_1}|dx + \cdots \int_{-\infty}^{t} |h(t-x)_{\underline{k}\cdot in}||f_n(x)|dx \qquad (6.8)$$

As in the previous theorems on Lagrange stability, each nonlinearity $N_{\underline{k}\cdot i}$ in the representation $R_{\underline{k}}$ is bounded, i.e.

$$|N_{\underline{k}\cdot i}| \le M_{\underline{k}\cdot i} \qquad i = 1,2,\cdots,n$$

The inequality in (6.8) may be further written as

$$|g(t)_{ss\cdot i}| \le M_{\underline{k}\cdot 1} \int_{-\infty}^{t} |h(t-x)_{\underline{k}\cdot i}|dx + \cdots + M_{\underline{k}\cdot n} \int_{-\infty}^{t} |h(t-x)_{\underline{k}\cdot in}|dx$$

where all the improper integrals are convergent because $\underline{k} \varepsilon \underline{I}$. Hence,

$$|g(t)_{ss\cdot i}| \le A_{\underline{k}\cdot i1} M_{\underline{k}\cdot 1} + A_{\underline{k}\cdot i2} M_{\underline{k}\cdot 2} + \cdots + A_{\underline{k}\cdot in} M_{\underline{k}\cdot n} \qquad (6.9)$$

Since the inequality (6.9) holds for each i, the result may be compactly expressed by the vector-matrix inequality

$$\underline{g}(t)_{ss} \underline{\le} [A_{\underline{k}}]\underline{M}_{\underline{k}} = \underline{B}_1 \qquad (6.10)$$

Thus, the waveforms in the system are bounded and $\underline{B}_1$ is a first approximation to the bound.

<u>Iteration of the Bound</u>

The vector bound $\underline{B}_1$ may be iterated to find a tighter bound $\underline{B}_2$ as described in the following steps:

(1) Determine $[A_{\underline{k}}]$ as a function of $\underline{k}$ for all $\underline{k} \in \underline{I}$. The matrix $[A_{\underline{k}}]$ is a function of n variables, namely, the n components of $\underline{k}$.

(2) For each $\underline{k} \in \underline{I}$, determine $\underline{M}_{\underline{k} \cdot \underline{B}_1}$, where the ith component may be found as

$$M_{\underline{k} \cdot \underline{B}_1 \cdot i} = \max_{z_i} |N_{\underline{k} \cdot i}| \qquad -B_{1 \cdot i} \leq z_i \leq B_{1 \cdot i}$$

(3) Determine $\underline{V}_{\underline{k} \cdot \underline{B}_1} = [A_{\underline{k}}]\underline{M}_{\underline{k} \cdot \underline{B}_1}$ as a function of $\underline{k}$.

(4) The second bound $\underline{B}_2$ is then found as

$$\underline{B}_2 = \min_{\underline{k}} \{\underline{V}_{\underline{k} \cdot \underline{B}_1}\}$$

where the minimization over $\underline{k}$ is performed for each component of $\underline{V}_{\underline{k} \cdot \underline{B}_1}$ separately.

(5) Repeat Steps (1) to (4) until $\underline{B}_{min}$ is attained. It should be noted that

$$\underline{V}_{\underline{k} \cdot \underline{B}_2} = [A_{\underline{k}}]\underline{M}_{\underline{k} \cdot \underline{B}_2}$$

where

$$M_{\underline{k}\cdot\underline{B}_2\cdot i} = \max_{zi}|N_{\underline{k}\cdot i}| \qquad -B_{2\cdot i} \le z_i \le B_{2\cdot i}$$

Then,

$$\underline{B}_3 = \min_{\underline{k}}\{V_{\underline{k}\cdot\underline{B}_2}\}$$

The following theorem is a second theorem on Lagrange stability. The sufficient conditions for boundedness are different from those postulated in Theorem I.d.

Theorem I.d'

The waveforms in the system of Figure 25 resulting from finite arbitrary conditions are bounded if

(i) Each of the linear blocks $H_{\underline{o}\cdot i}$ is stable, i.e., the poles of $H(s)_{\underline{o}\cdot i}$ are in the left half plane.

(ii) At least one nonlinearity $N_{\underline{o}\cdot m}$ is bounded, i.e., $|N(zm)_{\underline{o}\cdot m}| \le M_m$ for some i = m.

Proof: Considering the output $f(t)_{\underline{o}\cdot m}$ of the bounded nonlinearity $N_{\underline{o}\cdot m}$, it follows from hypothesis (ii) that

$$f(t)_{\underline{o}\cdot m} \le M_m \qquad \text{for all } t$$

The output of the linear block $H_{\underline{o}\cdot m}$ is

$$g(t)_m = g(t)_{trans\cdot m} + \int_{t_o}^{t} f_{\underline{o}\cdot m}(x)h_{\underline{o}\cdot m}(t-x)dx$$

In the steady state, one has

$$|g_m(t)| \le \int_{-\infty}^{t} |f_{\underline{o}\cdot m}(x)|\,|h_{\underline{o}\cdot m}(t-x)|dx$$

$$\le M_m \int_{-\infty}^{t} |h_{\underline{o}\cdot m}(t-x)|dx$$

The improper integral is finite because from hypothesis (i) the transfer function $H_{\underline{o}\cdot m}(s)$ is stable. Thus

$$|g_m(t)| \le M_m A_{\underline{o}\cdot m} \triangleq B_{1\cdot m}$$

The bound $B_{1\cdot m}$ on $g_m(t)$ may now be used to find a bound on $g_{m+1}(t)$. For this, one needs to define M_{m+1} as

$$M_{m+1} = \max_z |N_{\underline{o}\cdot m+1}(z)| \qquad -B_{1\cdot m} \le z \le B_{1\cdot m}$$

It follows that

$$|f_{m+1}(t)| \le M_{m+1}$$

and repeating the same steps for $g_{m+1}(t)$ as for $g_m(t)$ above, one has

$$|g_{m+1}(t)| \le M_{m+1} A_{\underline{o}\cdot m+1} \triangleq B_{1\cdot m+1}$$

Defining M_{m+2} as

$$M_{m+2} = \max_z |N(z)|_{\underline{o}\cdot m+2} \qquad -B_{1\cdot m+1} \leq z \leq B_{1\cdot m+1}$$

and repeating the same procedure for every block around the loop, one obtains the first bound $\underline{B}_1$:

$$|\underline{g}(t)_{ss}| \leq \underline{B}_1 = \begin{bmatrix} B_{1\cdot 1} \\ B_{1\cdot 2} \\ \vdots \\ B_{1\cdot n} \end{bmatrix}$$

<u>Iteration of the Bound</u>

If the procedure described above for determining the bound $\underline{B}_1$, it should be observed that the first component found was $B_{1\cdot m}$ and the last component $B_{1\cdot m-1}$. Define M_m' as

$$M_m' \triangleq \max_z |N_{\underline{o}\cdot m}(z)| \qquad -B_{1\cdot m-1} \leq z \leq B_{1\cdot m-1}$$

If $M_m' < M_m$, then the procedure may be repeated to get a tighter bound $\underline{B}_2$. The process is repeated until no improvement in the bound is obtained, i.e.,

$$\underline{B}_1 > \underline{B}_2 > \cdots > \underline{B}_{n-1} > \underline{B}_n = \underline{B}_{n+1} \triangleq \underline{B}_{min}$$

Remark

If the given system has more than one bounded nonlinearity, then the procedure of finding $\underline{B}_{min}$ can be modified to take advantage of the boundedness of the additional bounded nonlinearities. Denoting the bounded nonlinearities by $N_{m1}, N_{m2}, \cdots, N_{mr}$, the iteration of the bound may be effected in one of two ways:

(1) One may consider each bounded nonlinearity separately and thus obtain

$$\underline{B}_{min \cdot m_1}, \ \underline{B}_{min \cdot m_2}, \ \cdots, \ \underline{B}_{min \cdot m_r}$$

and then,

$$\underline{B}_{min} = \min\{\underline{B}_{min \cdot m_1}, \ \cdots, \ \underline{B}_{min \cdot m_r}\}$$

where the minimization is applied component-wise.

(2) Alternatively, the minimization may be applied at each step of the iteration. Thus, after finding

$$\underline{B}_{1 \cdot m_1}, \ \underline{B}_{1 \cdot m_2}, \ \cdots, \ B_{1 \cdot m_r}$$

$\underline{B}_1$ is obtained as

$$B_1 = \min\{\underline{B}_{1 \cdot m_2}, \cdots, B_{1 \cdot m_r}\}$$

and then the iteration is continued.

Both methods may be performed, and one may then select the least of the two bounds obtained.

Remark

The improvement criterion is valid for the system under consideration, and is referred to as Theorem II.d. Moreover, Theorem III.d on global asymptotic stability has the same statement and proof as for the corresponding previous theorems, namely, if $\underline{B}_{min} = 0$, then the system is asymptotically stable in the large.

Global Asymptotic Stability

This section presents two theorems on the global asymptotic stability of the system under consideration. The two theorems, namely, IV.d and V.d, may be regarded in some respects as extensions of Theorems IV and Theorems V. The analogy, however, is not quite complete. The main difference is that Theorems IV.d and V.d do not consider any equivalent representation $R_{\underline{k}}$ of the given system, but rather examine the system in its original representation only. The method of analysis which, in the previous theorems, was applicable to any of the system representations, could be adapted in the present theorems to the system in its original representation. The adaptation of the method of analysis, if at all possible, to equivalent representations of the system is recommended in the last chapter as a point for further investigation.

Theorem IV.d

The system in Figure 25 is asymptotically stable in the large, if it is stable in the sense of Lagrange, and if the following conditions are satisfied:

(1) Each linear block H_i is stable.

(2) Each nonlinearity $N_i(z)$ is located within a sector of the form $S_{-k_i}^{k_i}$ with $k_i > 0$.

(3) The numbers k_i satisfy the inequality

$$\prod_{1}^{n} K_i < \frac{1}{\prod_{1}^{n} A_i}$$

where

$$A_i = \int_{0}^{\infty} |h_i(t)|dt$$

Proof: Since the linear blocks H_i are stable, the transient components of the outputs $g_i(t)$ tend asymptotically to zero. Allowing the system sufficient time for the transient components to die out, one may consider the bounded steady state waveforms. The output of H_1 is

$$g_1(t) = \int_{-\infty}^{t} f_1(x)h_1(t-x)dx$$

Performing the same bounding steps as before, one has

$$|g_1(t)| \leq M_1A_1 \tag{6.11}$$

where

$$M_1 = \max_t |f_1(t)|$$

Denoting the maximum values of the waveforms $|g_i(t)|$ by E_i, i.e.

$$E_i = \max_t |g_i(t)|$$

it follows from hypothesis (2) that

$$M_1 < E_n k_1 \tag{6.12}$$

Substituting this bound on M_1 in the inequality (6.11), one has

$$|g_1(t)| \leq E_n k_1 A_1 \tag{6.13}$$

This relationship is valid for all time. In particular, it is valid for the instants of time when $|g_1(t)|$ attains its maximum value E_1, i.e.

$$E_1 \leq E_n k_1 A_1 \tag{6.14}$$

Repeating the same steps for $g_2(t)$, one obtains

$$g_2(t) = \int_{-\infty}^{t} f_2(x) h_2(t-x) dx$$

$$|g_2(t)| \leq M_2 A_2 \tag{6.15}$$

and from hypothesis (2), one has

$$M_2 < E_1 k_2 \tag{6.16}$$

Substituting this bound on M_2 in (6.15),

$$|g_2(t)| \leq E_1 k_2 A_2 \tag{6.17}$$

Substituting the bound on E_1 from (6.14) into the last inequality, one has

$$|g_2(t)| \leq E_n k_1 A_1 k_2 A_2 \tag{6.18}$$

In particular, (6.18) is valid when $|g_2(t)|$ takes its maximum value, i.e.

$$|E_2| \leq E_n k_1 A_1 k_2 A_2$$

Proceeding in this manner, one finally obtains

$$|g_n(t)| \leq E_n k_1 A_1 k_2 A_2 \cdots k_n A_n \tag{6.19}$$

and

$$|E_n| \leq E_n k_1 A_1 k_2 A_2 \cdots k_n A_n$$

The inequality in (6.20) may be written as

$$1 \leq \prod_{1}^{n} k_1 \prod_{1}^{n} A_i$$

or

$$\prod_{1}^{n} k_i \geq \frac{1}{\prod_{1}^{n} A_i} \tag{6.21}$$

The result in (6.21) is a contradiction to hypothesis (3). Therefore, the assumption of non-zero steady state waveforms cannot be true. Hence, the system is asymptotically stable in the large.

Theorem V(i).d

The system in Figure 25 is asymptotically stable in the large, if it is stable in the sense of Lagrange, and if the following conditions are satisfied (see Figure 27):

(1) All of the linear blocks H_i are stable.

(2) If one of the nonlinearities (say N_1) lies in the first and second quadrants and also lies within a sector

$$S_{-k_1}^{k_1} \qquad k_1 > 0$$

(3) Each of the other nonlinearities lies in a sector

$$S_{-k_i}^{k_i}, \quad k_i > 0 \qquad i = 1,2,\cdots,n$$

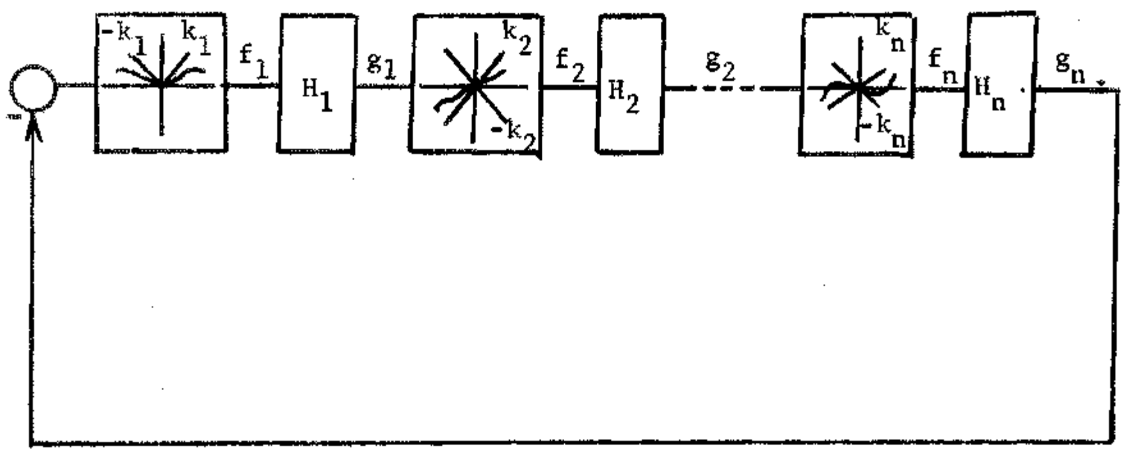

Figure 27. An Illustration of the Hypotheses of Theorem V(i).d

(4) The numbers k_i satisfy the inequality

$$k_1 k_2 k_3 \cdots k_n < \frac{1}{A_1^* A_2 A_3 \cdots A_n}$$

where

$$A_1^* = \max\{A_1^+, A_1^-\}$$

Proof: Assume that the bounded steady state waveforms are not identically zero. Considering the output $g_1(t)$ and writing $h_1(t) = h_1^+(t) + h_1^+(t)$, one has

$$g_1(t) = \int_{-\infty}^{t} f_1(x)[h_1^+(t-x) + h_1^-(t-x)]dx$$

$$g_1(t) = \int_{-\infty}^{t} f_1(x)h_1^+(t-x)dx + \int_{-\infty}^{t} f_1(x)h_1^-(t-x)dx$$

From hypothesis (2), it follows that $f_1(x) \geq 0$. Consequently, the first integral in (6.22) is positive and the second is negative. Let t_{max} denote an instant of time when $|g_1(t)|$ takes its maximum value,

$$E_1 = \max|g_1(t)| = |g_1(t_{max})| \neq 0$$

Two cases arise concerning the sign of $g_1(t_{max})$. The first case is that $g_1(t_{max}) > 0$. Let T_+ denote the set of time instants for which $g_1(t) > 0$. Evidently, $t_{max} \in T_+$. Equation (6.22) may be written as

$$g_1(t) = \int_{-\infty}^{t} f_1(x)h_1^+(t-x)dx + \int_{-\infty}^{t} f_1(x)h_1^-(t-x)dx > 0 \qquad t \in T_+$$

and since the second integral is non-positive, one has

$$g_1(t) \leq \int_{\infty}^{t} f_1(x)h_1^+(t-x)dx$$

Taking absolute values, one obtains

$$|g_1(t)| \leq \int_{-\infty}^{t} |f_1(x)||h_1^+(t-x)|dx$$

$$|g_1(t)| \leq M_1A_1^+ \quad \text{for all } t \in T_+ \tag{6.23}$$

and since $t_{max} \in T_+$, one has

$$|g_1(t_{max})| \leq M_1A_1^+$$

$$E_1 \leq M_1A_1^+ \tag{6.24}$$

The second case is $g_1(t_{max}) < 0$. Let T_- denote the set of time instants for which $g_1(t) < 0$. Evidently, $t_{max} \in T_-$. Equation (6.22) may be written as

$$g_1(t) = \int_{-\infty}^{t} f_1(x)h_1^+(t-x)dx + \int_{-\infty}^{t} f_1(x)h_1^-(t-x)dx$$

and since the first integral is nonnegative, one has

$$|g_1(t)| \leq \left|\int_{-\infty}^{t} f_1(x)h_1^-(t-x)dx\right|$$

and

$$g_1(t) \leq M_1A_1^- \quad \text{for all } t \in T_-$$

Since $t_{max} \in T_-$, one has

$$E_1 \leq M_1A_1^+ \tag{6.25}$$

It follows from (6.24) and (6.25) that

$$E_1 \leq M_1A_1^* \tag{6.26}$$

where

$$A_1^* = \max\{A_1^+, A_1^-\}$$

Starting with (6.26) and proceeding with the exact steps of Theorem IV.d, one finally obtains

$$|g_n(t)| \leq E_n k_1 A_1^* k_2 A_2 \cdots k_n A_n \qquad \text{for all } t$$

In particular the last inequality is true when $|g_n(t)|$ attains its maximum value E_n, i.e.

$$E_n \leq E_n(k_1k_2\cdots k_n)(A_1^*A_2\cdots A_n)$$

or

$$k_1k_2\cdots k_n \geq \frac{1}{A_1^*A_2\cdots A_n} \tag{6.27}$$

The result in (6.27) is a contradiction to the hypothesis in (4). Therefore, the initial assumption of non-zero steady state waveforms cannot be true, and the system is asymptotically stable in the large.

Theorem V(ii).d

The statement of this theorem is identical to that of the previous theorem except for hypothesis (2) which should read: If any of the nonlinearities lies in the intersection of the third and fourth quadrants with a sector

$$S_{-k_1}^{k_2}, \qquad k_1 > 0$$

The proof of this theorem is quite similar to that of Theorem V(i).d and will not be included here.

Example

In this example the stability of the system shown in Figure 28 is investigated. The system consists of a cascade of two nonlinearities and two linear blocks. The first nonlinearity N_1 is a full-wave rectifier and a limiter. The block H_1 is a first-order system which exhibits time delay. The second nonlinearity is a cubic characteristic, i.e.

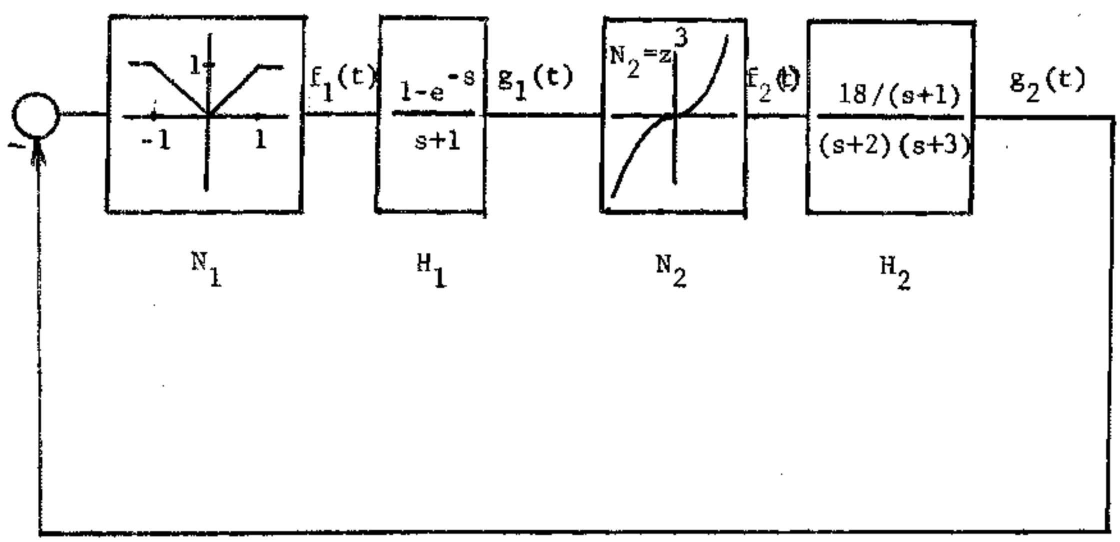

Figure 28. The System Discussed in the Example

$$N_2(z) = z^3$$

The second linear block H_2 is a third-order system. Because N_1 is bounded and H_1 and H_2 are stable, the system satisfies the conditions of Theorem I.d and, therefore, the steady state waveforms are bounded. It is shown in the following steps that no sustained oscillation occur in the system, i.e. the system is asymptotically stable in the large. First of all, one may find the required parameters. The impulse response of $H_1(s)$ is

$$h_1(t) = e^{-t} - u_{-1}(t-1)e^{-(t-1)}$$

Thus, $h_1(t)$ is positive on the interval (0,1) and negative on the interval $(1,\infty)$. Hence,

$$A_1^+ = \int_0^1 h_1(t)dt = \int_0^1 e^{-t}dt = 0.632$$

$$A_1^- = \int_1^\infty |h_1(t)| = \int_1^\infty |e^{-t} - e^{-(t-1)}| = 0.632$$

The impulse response of

$$h_2(t) = 9e^{-t} - 18e^{-2t} + 9e^{-3t}$$

and

$$A_2 = \int_0^\infty |h_2(t)|dt = 3$$

Next, one may find a bound $\underline{B}_1$ on $\underline{g}(t)$ as explained in Theorem I.d. In the steady state, one has

$$g_1(t) = \int_{-\infty}^{t} f_1(x)h_1(t-x)dx$$

Since $f_1(x) \geq 0$, one obtains

$$|g_1(t)| \leq M_1A_1^+ \qquad \text{whenever } g_1(t) > 0$$

and

$$|g_1(t)| \leq M_1A_1^- \qquad \text{whenever } g_1(t) < 0$$

with

$$M_1 = 1, \quad A_1^+ = 0.632, \quad \text{and} \quad A_1^- = 0.632$$

Thus,

$$|g_1(t)| \leq 0.632 \qquad \text{for all } t$$

Having found the bound on $g_1(t)$, a bound on $f_2(t)$ is found from the nonlinearity N_2, i.e.

$$|f_2(t)| \leq (0.632)^3 = 0.246$$

The bound on $g_2(t)$ is:

$$|g_2(t)| \le M_2A_2$$

$$|g_2(t)| \le 0.246 \times 3 = 0.738$$

Thus,

$$B_1 = \begin{bmatrix} 0.632 \\ 0.738 \end{bmatrix}$$

At this point one may use the improvement criterion (Theorem II.d) together with Theorem V(i).d. Over the intervals indicated by the bound $\underline{B}_1$, the system satisfies the conditions of Theorem V(i).d with

$$k_1 = 1, \quad k_2 = (0.632)^3/0.632 = 0.39$$

$$A_1^* = \max\{0.632, 0.632\} = 0.632, \quad A_2 = 3$$

The parameters k_1, k_2, A_1^*, A_2 satisfy the hypothesis

$$k_1k_2 \le \frac{1}{A_1^*A_2}$$

Therefore, the system is asymptotically stable in the large.

The bound $\underline{B}_1$ may be iterated as described in the section on the bound iteration for Theorem I'.d. The first five iterations result in the following values:

$$\underline{B}_1 = \begin{bmatrix} 0.632 \\ 0.738 \end{bmatrix}, \qquad \underline{B}_2 = \begin{bmatrix} 0.466 \\ 0.304 \end{bmatrix}, \qquad \underline{B}_3 = \begin{bmatrix} 0.19 \\ 0.022 \end{bmatrix}$$

$$\underline{B}_1 = \begin{bmatrix} 0.13\text{x}10^{-1} \\ 0.66\text{x}10^{-5} \end{bmatrix}, \quad \underline{B}_5 = \begin{bmatrix} 0.42\text{x}10^{-5} \\ 0.22\text{x}10^{-15} \end{bmatrix}, \quad \underline{B}_6 = \begin{bmatrix} 0.14\text{x}10^{-15} \\ 0.82\text{x}10^{-49} \end{bmatrix}$$

It is observed that the bounding is rapidly convergent, and the bound $\underline{B}_6$ is quite small. This is not unexpected in view of the fact that the system is asymptotically stable in the large. Instead of using Theorem V.d, one may establish asymptotic stability by showing that

$$\lim_{n\to\infty} \underline{B}_n = B_{min} = 0$$

This may be achieved by setting up a general expression for B_n and then taking the limit as $n \to \infty$.

In this chapter the method was adapted to a system with several nonlinear characteristics. Theorems I.d, II.d, and III.d were direct extensions of the previous results with the scalar parameters of the previous theorems generalized to vector and matrix parameters. In Theorems IV of the preceding chapters the scalar k representing the stability sector was related to the parameter A of the linear plant, but in this chapter the product of the parameters k_i representing the stability sectors, namely $\prod_1^n k_i$, was related to the product $\prod_1^n A_i$. The example illustrated the use of Theorem I.d in establishing the boundedness of the system and pointed out two different ways of verifying global asymptotic stability.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

This dissertation has presented a number of results on the stability of nonlinear feedback systems. The general method of approach adopted in this investigation is based on an indirect proof. Global asymptotic stability is verified by demonstrating that the assumption of non-zero steady-state motion leads to a contradiction.

The method was applied to stationary systems, time-varying systems, sampled-data systems, and multiple-nonlinearity systems. In each case five different results were obtained. The first result is a criterion that guarantees the Lagrange stability of the system and provides an upper bound on the amplitude of the steady-state motion (Theorems I). The statement of the theorem was basically the same for each of the types of systems considered; however, the process of iterating the steady-state bound had to be modified in the case of the system with time-varying nonlinearity. A second theorem on Lagrange stability was also presented for the multiple-nonlinearity system.

The second result is an improvement criterion that extends the domain of applicability of any existing criterion of the Popov type to include nonlinearities that otherwise could not be handled by that criterion (Theorems II). The theorem can also be used to improve any relevant results that may yet be developed.

The third result asserts that if the iteration of the steady-state bound has zero as a greatest lower bound, then the system is asymptotically stable in the large (Theorems III). This theorem may be used effectively to prove global asymptotic stability if a general expression for the nth iterated bound can be found and its limit as $n \to \infty$ can be evaluated. The iteration of the bound for any given nonlinearity may be conveniently realized on the digital computer.

The fourth and fifth results are theorems of the Popov type that guarantee global asymptotic stability by requiring the nonlinearity to be confined to a specified region of its input-output plane (Theorems IV and V). In Theorem IV, the stability region is a symmetrical sector of the input-output plane, whereas in Theorem V the stability region consists of two unequal semi-sectors. In certain cases, Theorem V could be more powerful than Theorem IV. This increased effectiveness might be gained, in some cases, at the expense of tighter constraints on the nonlinearity.

The results obtained have a broad domain of applicability. No significant restrictions are placed on the nature of the nonlinearity or the nonlinear plant. This accounts for the effectiveness of the results in handling certain systems which cannot be analyzed by other existing techniques.

Recommendations for Further Study

The method of approach of this research may be used to obtain additional results similar to those presented in this dissertation.

One possible extension is to attempt other techniques for bounding the forced component of $g(t)$. This may be possible if some restrictions are placed on the nature of the nonlinearity or the linear plant. This could lead to improved stability regions as in Theorems V(i) and V(ii).

It may be possible to make Theorem I more general by introducing two slope parameters, k_1 and k_2, instead of the single slope parameter k. The conditions on the nonlinearity could then be expressed in the following manner:

$$b_2 + k_1 z \le N_o(z) \le b_1 + k_1 z \qquad \text{for all } z > \lambda_1$$

$$b_4 + k_2 z \le N_o(z) \le b_3 + k_2 z \qquad \text{for all } z < \lambda_2$$

$$|N_o(z)| < C \qquad \text{for all } \lambda_2 \le z \le \lambda_1$$

where the other conditions remain unchanged.

In the results on time-varying systems in Chapter IV, the set $\tilde{I}$ was defined as being all of the values k for which the transient response of the $H_k(t)$ is stable and the absolute area of its impulse response is finite. It may be worthwhile to investigate the relationship between these two properties and to examine if, under any conditions, it is sufficient to require only one of the two properties to be true.

More work is needed to extend some of the results of Chapter VI to systems where the nonlinearities and the linear blocks occupy different relative positions in the overall system. Furthermore, additional work is needed to adapt or modify the analysis of Theorems IV.d and V.d so that the equivalent representations R_k can be handled. This would appreciably improve the applicability of the theorems.

This dissertation has presented some new ideas on the problem of stability in nonlinear feedback systems. The general method of analysis employed does not follow the pattern of previous investigations of the same problem. The results obtained do not exhaust the potentialities of the method.

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